Exponential laws for ultrametric partially differentiable functions and applications

Helge Glöckner

Abstract

We establish exponential laws for certain spaces of differentiable functions over a valued field $\mathbb{K}$. For example, we show that

$$C^{(\alpha,\beta)}(U \times V, E) \cong C^{\alpha}(U, C^{\beta}(V, E))$$

if $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$, $\beta \in (\mathbb{N}_0 \cup \{\infty\})^m$, $U \subseteq \mathbb{K}^n$ and $V \subseteq \mathbb{K}^m$ are open (or suitable more general) subsets, and E is a topological vector space. As a first application, we study the density of locally polynomial functions in spaces of partially differentiable functions over an ultrametric field (thus solving an open problem by Enno Nagel), and also global approximations by polynomial functions. As a second application, we obtain a new proof for the characterization of C^r-functions on $(\mathbb{Z}_p)^n$ in terms of the decay of their Mahler expansions. In both applications, the exponential laws enable simple inductive proofs via a reduction to the one-dimensional, vector-valued case.

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Introduction and statement of results

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1, \ldots, \alpha_n \in \mathbb{N}_0 \cup \{\infty\}$. A function $f : U \to \mathbb{R}$ on an open set $U \subseteq \mathbb{R}^n$ is called C^α if it admits continuous partial derivatives $\partial^\beta f : U \to \mathbb{R}$ for all multi-indices $\beta \in (\mathbb{N}_0)^n$ such that $\beta \leq \alpha$ (cf. [2]). In this article, we study an analogous notion of C^α-map $f : U \to E$ defined with the help of continuous extensions to certain partial difference quotient maps, for E a topological vector space over a topological field $\mathbb{K}$ and open subset $U \subseteq \mathbb{K}^n$. In fact, beyond open domains we consider C^α-functions on subsets $U \subseteq \mathbb{K}^n$ which are *locally cartesian* in the sense that each point in U has

a relatively open neighbourhood $V \subseteq U$ of the form $V = V_1 \times \cdots \times V_n$ for subsets $V_1, \ldots, V_n \subseteq \mathbb{K}$ without isolated points (see Definition 2.15). For $\mathbb{K}$ a complete ultrametric field and E an ultrametric Banach space, such functions have recently been introduced by E. Nagel [20]. Our (equivalent) definition differs from his in detail and avoids the use of spaces of linear operators (mimicking, instead, an approach to multi-variable C^r-maps pursued by De Smedt [7], Schikhof [21, §84] and the author [11, Definition 1.13]).

We endow the space $C^\alpha(U, E)$ of all E-valued C^α-maps on U with its natural vector topology (see Definition 3.1). The main result of this article is the following exponential law.

Theorem A. *Let $\mathbb{K}$ be a topological field, $n, m \in \mathbb{N}$, $U \subseteq \mathbb{K}^n$ and $V \subseteq \mathbb{K}^m$ be locally cartesian subsets, and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$, $\beta \in (\mathbb{N}_0 \cup \{\infty\})^m$. Then $f(x, \bullet) \in C^\beta(V, E)$ for each $f \in C^{(\alpha,\beta)}(U \times V, E)$, the map*

$$f^\vee \colon U \to C^\beta(V, E), \quad x \mapsto f(x, \bullet)$$

is C^α, and the mapping

$$\Phi \colon C^{(\alpha,\beta)}(U \times V, E) \to C^\alpha(U, C^\beta(V, E)), \quad f \mapsto f^\vee$$

is $\mathbb{K}$-linear and a topological embedding. If $\mathbb{K}$ is metrizable (e.g., if $\mathbb{K}$ is a valued field), then Φ is an isomorphism of topological $\mathbb{K}$-vector spaces.

Let $n \in \mathbb{N}$, $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$, $\mathbb{K}$ be a field and E be a $\mathbb{K}$-vector space. A function $p \colon U \to E$ on a subset $U \subseteq \mathbb{K}^n$ is called a *polynomial function of multidegree* $\leq \alpha$ if there exist $a_\beta \in E$ for multi-indices $\beta \in (\mathbb{N}_0)^n$ with $\beta \leq \alpha$ such that $a_\beta = 0$ for all but finitely many β and

$$p(x) = \sum_{\beta \leq \alpha} x^\beta a_\beta \quad \text{for all } x = (x_1, \ldots, x_n) \in U,$$

with $x^\beta := x_1^{\beta_1} \cdot \ldots \cdot x_n^{\beta_n}$. We write $\mathrm{Pol}(U, E)$ for the space of all E-valued polynomial functions on U. If $(\mathbb{K}, |.|)$ is a valued field, $U \subseteq \mathbb{K}^n$ a subset and E a topological $\mathbb{K}$-vector space, we say that a function $f \colon U \to E$ is *locally polynomial* of multidegree $\leq \alpha$ if each $x \in U$ has an open neighbourhood V in U such that $f|_V = p$ for some polynomial function $p \colon \mathbb{K}^n \supseteq V \to E$ of multidegree $\leq \alpha$. We write $\mathrm{LocPol}_{\leq \alpha}(U, E)$ for the space of all locally polynomial E-valued functions of multidegree $\leq \alpha$ on U. Using Theorem A

to perform a reduction to the one-dimensional case (settled for scalar-valued functions as a special case of [20, Proposition II.40]), we obtain:

Theorem B. *For every complete ultrametric field* $\mathbb{K}$*, compact cartesian subset* $U \subseteq \mathbb{K}^n$*, locally convex topological* $\mathbb{K}$*-vector space* E *and* $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$*, the space* $\mathrm{LocPol}_{\leq \alpha}(U, E)$ *of* E*-valued locally polynomial functions of multidegree* $\leq \alpha$ *is dense in* $C^\alpha(U, E)$*.*

For $E = \mathbb{K}$ and $n \geq 2$, this answers a question raised by E. Nagel in the case of integer exponents (he also considered fractional differentiability; the case of $\alpha \in [0, \infty[^n \setminus \mathbb{N}_0^n$ remains open).[1]

Likewise, the exponential law can be used to reduce questions concerning the Mahler expansions of functions of several variables to the familiar case of single-variable functions. We obtain the following result (the scalar-valued case of which was published earlier in the independent work [20], based on a different strategy of proof which exploits topological tensor products):[2]

Theorem C. *Let* E *be a sequentially complete locally convex space over* $\mathbb{Q}_p$ *and* $f\colon (\mathbb{Z}_p)^n \to E$ *be a continuous function, where* $n \in \mathbb{N}$*. Let*

$$f(x) = \sum_{\nu \in \mathbb{N}_0^n} \begin{pmatrix} x \\ \nu \end{pmatrix} a_\nu$$

be the Mahler expansion of f *with the Mahler coefficients* $a_\nu \in E$*. Let* $\alpha \in \mathbb{N}_0^n$*. Then* f *is* C^α *if and only if*

$$q(a_\nu)\nu^\alpha \to 0 \quad \text{as } |\nu| \to \infty \tag{1}$$

for each continuous ultrametric seminorm q *on* E*, where* $|\nu| := \nu_1 + \cdots + \nu_n$*. Given* $r \in \mathbb{N}_0$*, the map* f *is* C^r *if and only if*

$$q(a_\nu)|\nu|^r \to 0 \quad \text{as } |\nu| \to \infty,$$

for each q *as before.*

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[1] Personal communication at the 12th International Conference on p-Adic Functional Analysis, Winnipeg, July 2012.

[2] Compare also [7, theorem on p. 140] for the case of scalar-valued C^1-functions on $\mathbb{Z}_p \times \mathbb{Z}_p$, established by direct calculations.

Passage to the Mahler coefficients allows $C^\alpha(\mathbb{Z}_p^n, E)$ to be identified with a suitable weighted E-valued c_0-space, whose weight is modelling the decay condition (1) (see Proposition 6.11 for details).

Here, we use that a function $f\colon \mathbb{K}^n \supseteq U \to E$ is C^r if and only if it is C^α for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq r$. To generalize this idea, let us return to a topological field $\mathbb{K}$ and topological $\mathbb{K}$-vector space E. Let $n_1, \ldots, n_\ell \in \mathbb{N}$, $n := n_1 + \cdots + n_\ell$ and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$. Agreeing to consider $\mathbb{K}^n$ as the direct product $\mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_\ell}$, we say that a function $f\colon U \to E$ on a locally cartesian subset $U \subseteq \mathbb{K}^n$ is C^α if it is C^β for all $\beta \in \mathbb{N}_0^n$ such that $|\beta_j| \leq \alpha_j$ for all $j \in \{1, \ldots, \ell\}$, where we wrote $\beta = (\beta_1, \ldots, \beta_\ell)$ according to the decomposition $\mathbb{N}_0^n = \mathbb{N}_0^{n_1} \times \cdots \times \mathbb{N}_0^{n_\ell}$. Then again an exponential law holds.

Theorem D. *Let $\mathbb{K}$ be a Hausdorff topological field, $n, m \in \mathbb{N}$ and $U \subseteq \mathbb{K}^n$ as well as $V \subseteq \mathbb{K}^m$ be locally cartesian subsets. Fix decompositions $n = n_1 + \cdots + n_\ell$ and $m = m_1 + \cdots + m_k$ with $n_1, \ldots, n_\ell, m_1, \ldots, m_k \in \mathbb{N}$. Let $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$ and $\beta \in (\mathbb{N}_0 \cup \{\infty\})^k$. Then $f(x, \bullet) \in C^\beta(V, E)$ for each $f \in C^{(\alpha,\beta)}(U \times V, E)$, the map*

$$f^\vee\colon U \to C^\beta(V, E), \quad x \mapsto f(x, \bullet)$$

is C^α, and the mapping

$$\Phi\colon C^{(\alpha,\beta)}(U \times V, E) \to C^\alpha(U, C^\beta(V, E)), \quad f \mapsto f^\vee$$

is $\mathbb{K}$-linear and a topological embedding. If $\mathbb{K}$ is metrizable (e.g., if $\mathbb{K}$ is a valued field), then Φ is an isomorphism of topological $\mathbb{K}$-vector spaces.

As a consequence, reducing to the case of C^r-maps on compact cartesian sets treated (in the scalar-valued case) in [20, Proposition II.40], we obtain the density of suitable locally polynomial functions in $C^\alpha(U, E)$ for all locally closed, locally cartesian sets $U \subseteq \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_\ell}$ and $\alpha \in \mathbb{N}_0^\ell$ (Proposition 9.11). Generalizing the case of scalar-valued functions on compact cartesian sets treated in [20, Corollary II.42], we prove the density of polynomial functions in many cases. Taking $\ell = 1$ and $\ell = n$, respectively, our Proposition 9.12 subsumes:

Theorem E. *Let $\mathbb{K}$ be a complete ultrametric field, E be a locally convex topological $\mathbb{K}$-vector space, $n \in \mathbb{N}$ and $U \subseteq \mathbb{K}^n$ be a locally closed, locally*

cartesian subset. Then the space $\mathrm{Pol}(U,E)$ *of all E-valued polynomial functions on U is dense in $C^r(U,E)$, for each $r \in \mathbb{N}_0 \cup \{\infty\}$. Moreover,* $\mathrm{Pol}(U,E)$ *is dense in $C^\alpha(U,E)$, for each $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$.*

Every open subset $U \subseteq \mathbb{K}^n$ satisfies the hypotheses of Theorem E, and also every locally cartesian subset which is closed or locally compact.

Polynomial approximations to functions of a single variable were already studied in [4].

As a special case, taking $\ell = 2$ the preceding approach subsumes a notion of $C^{(r,s)}$-maps on locally cartesian subsets $U \subseteq \mathbb{K}^n \times \mathbb{K}^m$. For a corresponding notion of $C^{(r,s)}$-maps on open subsets $U \subseteq \mathbb{R}^n \times \mathbb{R}^m$ (or, more generally, on open subset $U \subseteq E_1 \times E_2$ with real locally convex spaces E_1 and E_2), the reader is referred to the earlier works [3] and [2], where also exponential laws for the corresponding function spaces (similar to Theorem D) can be found. Special cases (and variants) of such $C^{(r,s)}$-maps have been used in many parts of analysis (see, e.g., [1] for analogues of $C^{(0,r)}$-maps on real Banach spaces based on continuous Fréchet differentiability; [10, 1.4] for $C^{(0,r)}$-maps between real locally convex spaces; [12] for $C^{(r,s)}$-maps on finite-dimensional real domains; and [9, p. 135] for certain $\mathrm{Lip}^{(r,s)}$-maps in the convenient setting of analysis (a backbone of which are exponential laws for spaces of smooth functions, see also [17]). We mention that exponential laws for suitable spaces of *smooth* functions over locally compact topological fields (also in infinite dimensions) were already established in [13, Propositions 12.2 and 12.6]. The possibility of exponential laws for $C^{r,s}$-maps was conjectured there [13, p. 10].

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1 Preliminaries and notation

All topological fields occurring in this article are assumed Hausdorff and non-discrete; all topological vector spaces are assumed Hausdorff. A *valued field* is a field $\mathbb{K}$, equipped with an absolute value $|.|\colon \mathbb{K} \to [0,\infty[$ which defines a non-discrete topology on $\mathbb{K}$. If $|.|$ satisfies the ultrametric inequality, we call $(\mathbb{K}, |.|)$ an *ultrametric field*. In this case, we call $\mathbb{O} := \{z \in \mathbb{K}\colon |z| \leq 1\}$ the *valuation ring* of $\mathbb{K}$. A topological vector space E over an ultrametric field

$(\mathbb{K}, |.|)$ is called *locally convex* if the set of all open $\mathbb{O}$-submodules of E is a basis for the filter of 0-neighbourhoods in E (we then simply call E a *locally convex space*). Equivalently, E is locally convex if its topology is defined by a set of seminorms $q\colon E \to [0, \infty[$ which are ultrametric in the sense that $q(x+y) \leq \max\{q(x), q(y)\}$ for all $x, y \in E$ (see [18], [22] and [23] for further information on such spaces). A valued field $(\mathbb{K}, |.|)$ is called *complete* if the metric defined via $d(x, y) := |x - y|$ makes $\mathbb{K}$ a complete metric space. If X is a topological space, $Y \subseteq X$ a subset and $U \subseteq Y$ a relatively open subset, we usually simply say that "U is open in Y" or "$U \subseteq Y$ is open". In the rare cases when U is intended to by open in X, we shall say so explicitly. As usual, by a "clopen" set we mean a set which is both closed and open. If $\mathbb{K}$ is a topological field, we always endow $\mathbb{K}^n$ with the product topology. A subset Y of a topological space X is called *locally closed* if, for each $x \in Y$ and neighbourhood $U \subseteq Y$ (with respect to the induced topology) there exists a neighbourhood $V \subseteq U$ of x which is closed in X. For example, every closed subset of a regular topological space (like $\mathbb{K}^n$) is locally closed, and also every open subset. Locally compact subsets of a topological space are locally closed as well. If (X, d) is a metric space, $x \in X$ and $r > 0$, we write $B^d_r(x) := \{y \in X\colon d(x, y) < r\}$ and $\overline{B}^d_r(x) := \{y \in X\colon d(x, y) \leq r\}$ for the open and closed balls, respectively. Given a non-empty subset $Y \subseteq X$, we set $B^d_r(Y) := \{x \in X\colon (\exists y \in Y)\, d(x, y) < r\} = \bigcup_{y \in Y} B^d_r(y)$. If $(E, \|.\|)$ is a normed space over a valued field $(\mathbb{K}, |.|)$, $x \in \mathbb{K}$ and $r > 0$, we use the notation $B^E_r(x) := \{y \in E\colon |x - y|) < r\}$. We write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

If X is a set and $(f_j)_{j \in J}$ a family of mappings $f_j\colon X \to X_j$ to topological spaces X_j, then the set of all pre-images $f_j^{-1}(U)$ (with $j \in J$ and U open in X_j) is a subbasis for a topology $\mathcal{O}$ on X, called the *initial topology* with respect to the family $(f_j)_{j \in J}$. For any topological space Y, a map

$$f\colon Y \to (X, \mathcal{O})$$

is continuous if and only if $f_j \circ f$ is continuous for each $j \in J$. The topology $\mathcal{O}$ is unchanged if we take U only in a basis of open subsets of X_j, or in a subbasis. This readily implies the following well-known fact (the "transitivity of initial topologies"), which will be used frequently:

Lemma 1.1 *Let X be a topological space whose topology $\mathcal{O}$ is initial with respect to a family $(f_j)_{j \in J}$ of mappings $f_j\colon X \to X_j$ to topological spaces X_j.*

Assume that the topology on X_j is initial with respect to a family $(f_{i,j})_{i \in I_j}$ of mappings $f_{i,j}\colon X_j \to X_{i,j}$ to topological spaces $X_{i,j}$. Then $\mathcal{O}$ is also the initial topology with respect to the family $(f_{i,j} \circ f_j)_{j \in J, i \in I_j}$ of compositions $f_{i,j} \circ f_j \colon X \to X_{i,j}$. □

A mapping $f\colon X \to Y$ between topological spaces is called a *topological embedding* (or simply: an embedding) if it is a homeomorphism onto its image. This holds if and only if f is injective and the topology on X is initial with respect to f.

2 C^α-maps on subsets of $\mathbb{K}^n$

We define C^α-maps and record elementary properties of such maps and their domains of definition. The discussion follows the treatment of C^k-maps in [11, 2.5–2.16] so closely, that some proofs can be replaced by references to [11].

2.1 Throughout this section, let $\mathbb{K}$ be a topological field, E be a topological $\mathbb{K}$-vector space, $n \in \mathbb{N}$ and $U \subseteq \mathbb{K}^n$ be a subset (where $\mathbb{K}^n$ is equipped with the product topology).

2.2 We say that U *cartesian* if it is of the form $U = U_1 \times \cdots \times U_n$, for some subsets $U_1, \ldots, U_n$ of $\mathbb{K}$ without isolated points.[3] If every point in U has a relatively open neighbourhood in U which is cartesian, then U is called *locally cartesian*.[4]

2.3 Observe that if $U \subseteq \mathbb{K}^n$ is a locally cartesian subset, then each $x \in U$ has an open cartesian neighbourhood $W \subseteq U$ of the form $W = U \cap Q$, for some open *cartesian* subset $Q \subseteq \mathbb{K}^n$ (which is sometimes useful).

[Indeed, $x = (x_1, \ldots, x_n) \in U$ has a relatively open neighbourhood $V \subseteq U$ of the form $V = V_1 \times \cdots \times V_n$, for some subsets $V_1, \ldots, V_n \subseteq \mathbb{K}$ without isolated points. Thus, there is an open subset $Y \subseteq \mathbb{K}^n$ such that $U \cap Y = V_1 \times \cdots \times V_n$. If $Q_1, Q_2, \ldots, Q_n$ are open neighbourhoods of $x_1, \ldots, x_n$ in $\mathbb{K}$, respectively, such that $Q := Q_1 \times \cdots \times Q_n \subseteq Y$, then

$$U \cap Q = U \cap Y \cap Q = V \cap Q = (Q_1 \times V_1) \times \cdots \times (Q_n \cap V_n), \qquad (2)$$

[3] Compare [11, Remark 2.16].

[4] In the case of a complete ultrametric field, such a set is called a "locally cartesian set whose factors contain no isolated points" in [20].

where $V_1 \cap Q_1, \ldots, V_n \cap Q_n$ do not have isolated points.]

Note that, if $P \subseteq U$ is a given neighbourhood of x, we can choose Q so small that $U \cap Q \subseteq P$.

2.4 As usual, for $i \in \{1, \ldots, n\}$ we set $e_i := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{K}^n$, with i-th entry 1. Given a "multi-index" $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| := \sum_{i=1}^n \alpha_i$ and $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$. If $\alpha, \beta \in (\mathbb{N}_0 \cup \{\infty\})^n$, we say that $\alpha \leq \beta$ if $\alpha_j \leq \beta_j$ for all $j \in \{1, \ldots, n\}$. We define $\alpha + \beta$ component-wise, with $r + \infty = \infty + r := \infty$ for all $r \in \mathbb{N}_0 \cup \{\infty\}$.

2.5 For $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$, our definition of a C^α-map $f\colon U \to E$ on a cartesian set U will involve a certain continuous extension $f^{<\beta>}$ to an iterated partial difference quotient map $f^{>\beta<}$ corresponding to each multi-index $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$. It is convenient to define the domains $U^{<\beta>}$ and $U^{>\beta<}$ of these mappings first. They will be subsets of $\mathbb{K}^{n+|\beta|}$. It is useful to write elements $x \in \mathbb{K}^{n+|\beta|}$ in the form $x = (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$, where $x^{(i)} \in \mathbb{K}^{1+\beta_i}$ for $i \in \{1, \ldots, n\}$. We write $x^{(i)} = (x_0^{(i)}, x_1^{(i)}, \ldots, x_{\beta_i}^{(i)})$ with $x_j^{(i)} \in \mathbb{K}$ for $j \in \{0, \ldots, \beta_i\}$.

2.6 Let $U \subseteq \mathbb{K}^n$ be locally cartesian. Given $\beta \in \mathbb{N}_0^n$, we define $U^{<\beta>}$ as the set of all $x \in \mathbb{K}^{n+|\beta|}$ such that, for all $i_1 \in \{0, 1, \ldots, \beta_1\}, \ldots, i_n \in \{0, 1, \ldots, \beta_n\}$, we have

$$(x_{i_1}^{(1)}, \ldots, x_{i_n}^{(n)}) \in U\,.$$

We let $U^{>\beta<}$ be the set of all $x \in U^{<\beta>}$ such that, for all $i \in \{1, \ldots, n\}$ and $0 \leq j < k \leq \beta_i$, we have $x_j^{(i)} \neq x_k^{(i)}$.

Example 2.7 If $U \subseteq \mathbb{K}^n$ is cartesian, say $U = U_1 \times \cdots \times U_n$ with subsets $U_i \subseteq \mathbb{K}$, then simply

$$U^{<\beta>} \;=\; U_1^{1+\alpha_1} \times U_2^{1+\beta_2} \times \cdots \times U_n^{1+\beta_n}\,. \tag{3}$$

2.8 It is easy to see that $U^{>\beta<}$ is an open subset of $U^{<\beta>}$. If U is cartesian, it is also easy to check that $U^{>\beta<}$ is dense in $U^{<\beta>}$.

Remark 2.9 Unfortunately, $U^{>\beta<}$ need *not* be dense in $U^{<\beta>}$ if $U \subseteq \mathbb{K}^n$ is merely locally cartesian. For this reason, we shall define C^α-maps by a reduction to the case of cartesian subsets. Also the definition of the topology on function spaces is made slightly more complicated by the fact that continuous extensions $f^{<\beta>}$ cannot be defined globally in general (only after restriction of f to cartesian subsets).

Example 2.10 Let $p \geq 3$ be a prime and V_2 and W_2 be compact, open, non-empty, disjoint subsets of $\mathbb{Q}_p$. Pick $v \in V_2$, $w \in W_2$. Let

$$V_1 := \left\{ \sum_{k=0}^{\infty} a_k p^k \colon (a_k)_{k \in \mathbb{N}_0} \in \{0,1\}^{\mathbb{N}_0} \right\}$$

and $W_1 := \{\sum_{k=0}^{\infty} a_k p^k \colon (a_k)_{k\in\mathbb{N}_0} \in \{0,2\}^{\mathbb{N}_0}\}$. Then V_1 and W_1 are compact subsets of $\mathbb{Q}_p$ without isolated points, and thus

$$U := (V_1 \times V_2) \cup (W_1 \times W_2)$$

is a compact, locally cartesian subset of $\mathbb{Q}_p \times \mathbb{Q}_p$. Note that $(0,0,v,w) \in U^{<(1,1)>}$ (because $(0,v),(0,w) \in U$), and that

$$Q := (\mathbb{Q}_p \times \mathbb{Q}_p \times V_2 \times W_2) \cap U^{<1,1>}$$

is an open neighbourhood of $(0,0,v,w)$ in $U^{<1,1>}$. One can show that $Q \cap U^{>(1,1)<} = \emptyset$. Hence $(0,0,v,w)$ is not in the closure of $U^{>(1,1)<}$.

[For the proof, we use that $(V_1 \times V_2) \cap (W_1 \times W_2) = \emptyset$ and $V_1 \cap W_1 = \{0\}$. Suppose there exists $(x,y,v_2,w_2) \in Q \cap U^{>(1,1)<}$. Then $x \neq y$, $v_2 \neq w_2$ and $v_2 \in V_2$, $w_2 \in W_2$. Note that $(x,v_2),(x,w_2) \in U$ forces $x \in V_1 \cap W_1 = \{0\}$. Likewise, $(y,v_2),(y,w_2) \in U$ entails that $y = 0$. Hence $x = y$, contradiction. Hence (x,y,v_2,w_2) cannot exist.]

Remark 2.11 A simple induction on $|\beta|$ shows that the sets $U^{<\beta>}$ can be defined alternatively by recursion on $|\beta|$, as follows: Set $U^{<0>} := U$. Given $\beta \in \mathbb{N}_0^n$ such that $|\beta| \geq 1$, pick $\gamma \in \mathbb{N}_0^n$ such that $\beta = \gamma + e_i$ for some $i \in \{1,\ldots,n\}$. Then $U^{<\beta>}$ is the set of all elements $x \in \mathbb{K}^{n+|\beta|}$ such that $(x^{(1)},\ldots,x^{(i-1)},x_0^{(i)},x_1^{(i)},\ldots,x_{\beta_i-1}^{(i)},x^{(i+1)},\ldots,x^{(n)}) \in U^{<\gamma>}$ holds as well as $(x^{(1)},\ldots,x^{(i-1)},x_{\beta_i}^{(i)},x_1^{(i)},\ldots,x_{\beta_i-1}^{(i)},x^{(i+1)},\ldots,x^{(n)}) \in U^{<\gamma>}$.

We now define certain mappings $f^{>\beta<} \colon U^{>\beta<} \to E$ and show afterwards that they can be interpreted as partial difference quotient maps.

Definition 2.12 We set $f^{>0<} := f$. Given a multi-index $\beta \in \mathbb{N}_0^n$ such that $|\beta| \geq 1$, we define $f^{>\beta<}(x)$ as the sum

$$\sum_{j_1=0}^{\beta_1} \cdots \sum_{j_n=0}^{\beta_n} \left(\prod_{k_1 \neq j_1} \frac{1}{x_{j_1}^{(1)} - x_{k_1}^{(1)}} \cdot \ldots \cdot \prod_{k_n \neq j_n} \frac{1}{x_{j_n}^{(n)} - x_{k_n}^{(n)}} \right) f(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)}) \quad (4)$$

for $x \in U^{>\beta<}$, using the notational conventions from **2.5**. The products are taken over all $k_\ell \in \{0, \ldots, \beta_\ell\}$ such that $k_\ell \neq j_\ell$, for $\ell \in \{1, \ldots, n\}$ (and empty products are defined as the element $1 \in \mathbb{K}$).

The map $f^{>\beta<}$ has important symmetry properties.

Lemma 2.13 *Assume that $\beta \in \mathbb{N}_0^n$, $i \in \{1, \ldots, n\}$ and π is a permutation of $\{0, 1, \ldots, \beta_i\}$. Then $(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_{\pi(0)}, \ldots, x^{(i)}_{\pi(\beta_i)}, x^{(i+1)}, \ldots, x^{(n)}) \in U^{>\beta<}$ for each $x \in U^{>\beta<}$, and*

$$f^{>\beta<}(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_{\pi(0)}, \ldots, x^{(i)}_{\pi(\beta_i)}, x^{(i+1)}, \ldots, x^{(n)}) \;=\; f^{>\beta<}(x)\,. \tag{5}$$

Proof. The proof of [11, Lemma 2.11] can be repeated verbatim. $\square$

The following lemma shows that $f^{>\beta<}$ can indeed be interpreted as a partial difference quotient map.

Lemma 2.14 *For each $i \in \{1, \ldots, d\}$ and $x \in U^{>e_i<}$, the element $f^{>e_i<}(x)$ is given by*

$$\frac{f(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_0, x^{(i+1)}, \ldots, x^{(n)}) - f(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_1, x^{(i+1)}, \ldots, x^{(n)})}{x^{(i)}_0 - x^{(i)}_1}.$$

If $\beta \in \mathbb{N}_0^n$ such that $|\beta| \geq 2$, let $\gamma \in \mathbb{N}_0^n$ be a multi-index such that $\beta = \gamma + e_i$ for some $i \in \{1, \ldots, n\}$. Then $f^{>\beta<}(x)$ is given by

$$\begin{aligned}\frac{1}{x^{(i)}_0 - x^{(i)}_{\beta_i}} \cdot \Big(& f^{>\gamma<}(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_0, x^{(i)}_1, \ldots, x^{(i)}_{\beta_i - 1}, x^{(i+1)}, \ldots, x^{(n)}) \\ & - f^{>\gamma<}(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_{\gamma_i}, x^{(i)}_1, \ldots, x^{(i)}_{\beta_i - 1}, x^{(i+1)}, \ldots, x^{(n)})\Big)\end{aligned} \tag{6}$$

for all $x \in U^{>\beta<}$.

Proof. The proof of [11, Lemma 2.12] carries over verbatim. $\square$

Definition 2.15 Let $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$, E be a topological $\mathbb{K}$-vector space and $U \subseteq \mathbb{K}^n$ be a subset.

(a) If U is cartesian, we say that a function $f\colon U \to E$ is C^α if the mapping $f^{>\beta<}\colon U^{>\beta<} \to E$ admits a continuous extension $f^{<\beta>}\colon U^{<\beta>} \to E$, for each $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$.

(b) If U is locally cartesian, we say that f is C^α if $f|_V$ is C^α, for each cartesian relatively open subset $V \subseteq U$.

Remark 2.16 Since $U^{>\beta<}$ is dense in $U^{<\beta>}$, the continuous extension $f^{<\beta>}$ of $f^{>\beta<}$ in Definition 2.15 (a) is unique whenever it exists.

Remark 2.17 Assume that U is cartesian and $f\colon U \to E$ is C^α in the sense of Definition 2.15 (a). If $V \subseteq U$ is a cartesian open subset, then $f^{<\beta>}|_{V^{<\beta>}}$ provides a continuous extension for $(f|_V)^{>\beta<}$. Hence f is C^α also in the sense of Definition 2.15 (b), with $(f|_V)^{<\beta>} = f^{<\beta>}|_{V^{<\beta>}}$.

Remark 2.18 If $f\colon U \to E$ is C^α, then f is continuous,

[In fact, $f|_V = (f|_V)^{<0>}$ is continuous for each cartesian open subset $V \subseteq U$. As the latter form an open cover of U, the assertion follows.]

Remark 2.19 If $f\colon U \to E$ is C^α, then f is also C^β for each $\beta \in (\mathbb{N}_0 \cup \{\infty\})^n$ such that $\beta \leq \alpha$. This is immediate from the definition.

We readily deduce from Lemma 2.13:

Lemma 2.20 *Let $f\colon U \to E$ be a C^α-map on a cartesian subset $U \subseteq \mathbb{K}^n$, $\beta \in \mathbb{N}_0^n$ with $\beta \leq \alpha$, $i \in \{1, \ldots, n\}$ and π be a permutation of $\{0, 1, \ldots, \beta_i\}$. Then*

$$(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_{\pi(0)}, \ldots, x^{(i)}_{\pi(\beta_i)}, x^{(i+1)}, \ldots, x^{(n)}) \;\in\; U^{<\beta>} \tag{7}$$

for each $x \in U^{<\beta>}$, and

$$f^{<\beta>}(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_{\pi(0)}, \ldots, x^{(i)}_{\pi(\beta_i)}, x^{(i+1)}, \ldots, x^{(n)}) \;=\; f^{<\beta>}(x)\,. \tag{8}$$

Proof. The proof of [11, Lemma 2.14] can be repeated verbatim. $\square$

The following variant of our Lemma 2.14 is available for $f^{<\beta>}$:

Lemma 2.21 *Let $f\colon U \to E$ be a C^α-map on a cartesian subset $U \subseteq \mathbb{K}^n$ and $\beta, \gamma \in \mathbb{N}_0^n$ such that $\beta = \gamma + e_i$ for some $i \in \{1, \ldots, n\}$, and $\beta \leq \alpha$. Then $f^{<\beta>}(x)$ is given by*

$$\frac{1}{x^{(i)}_0 - x^{(i)}_{\beta_i}} \cdot \Big(f^{<\gamma>}(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_0, x^{(i)}_1, \ldots, x^{(i)}_{\beta_i-1}, x^{(i+1)}, \ldots, x^{(n)}) \\ - f^{<\gamma>}(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_{\beta_i}, x^{(i)}_1, \ldots, x^{(i)}_{\beta_i-1}, x^{(i+1)}, \ldots, x^{(n)}) \Big) \tag{9}$$

for all $x \in U^{<\beta>}$ such that $x^{(i)}_0 \neq x^{(i)}_{\beta_i}$.

Proof. The proof of [11, Lemma 2.15] can be repeated verbatim. □

Remark 2.22 If $U \subseteq \mathbb{K}^n$ is locally cartesian and $P \subseteq U$ an open subset, then also P is locally cartesian.

[Proof. Let $x \in U$. We have seen at the end of 2.3 that there exists an open cartesian neighbourhood $U \cap Q$ of x in U that is contained in P.]

Remark 2.23 Let $U \subseteq \mathbb{K}^n$ be locally cartesian and $W \subseteq U$ be an open subset. Since every cartesian open subset of W is also a cartesian open subset of U, it is clear that $f|_W$ is C^α for each C^α-map $f\colon U \to E$.

Frequently, differentiability properties that are defined via global continuous extensions of difference quotient maps are, nonetheless, local properties.[5] Also being C^α is a local property.

Lemma 2.24 *Let $f\colon U \to E$ be a mapping on a locally cartesian subset $U \subseteq \mathbb{K}^n$. Assume that each $x \in U$ has an open cartesian neighbourhood W in U such that $f|_W$ is C^α. Then f is C^α.*

Proof. The proof is presented in a form which can be re-used later. We show that $f|_V$ is C^α for each open cartesian subset $V \subseteq U$.

Construction of $(f|_V)^{<\beta>}$. We show that $(f|_V)^{>\beta<}$ has a continuous extension $(f|_V)^{<\beta>}$, for each $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$. The proof is by induction on $|\beta|$. If $\beta = 0$ and each point has a neighbourhood on which f is continuous, then $(f|_V)^{<0>} = f|_V$ is continuous. Now assume that $|\beta| \geq 1$. Set $I_\beta := \{i \in \{1, \ldots, n\}\colon \beta_i > 0\}$. For $i \in I_\beta$ and $j < k$ in $\{0, 1, \ldots, \beta_i\}$, define

$$D_{\beta,i,j.k} := \{x \in V^{<\beta>}\colon x_j^{(i)} \neq x_k^{(i)}\}\,.$$

Then $V^{>\beta<} \subseteq D_{\beta,i,j,k}$, and $D_{\beta,i,j.k}$ is an open and dense set in $V^{<\beta>}$. Let $\beta_i' := \beta - e_i$. By induction, continuous maps $(f|_V)^{<\gamma>}\colon V^{<\gamma>} \to E$ extending $(f|_V)^{>\gamma<}$ exist for all $\gamma \in \mathbb{N}_0^n$ such that $\gamma \leq \alpha$ and $|\gamma| < |\beta|$. In particular, $(f|_V)^{<\beta_i'>}$ exists, enabling us to define continuous maps $f_{\beta,i,j,k}\colon D_{\beta,i,j,k} \to E$ sending $x \in D_{\beta,i,j,k}$ to

$$\frac{1}{x_j^{(i)} - x_k^{(i)}}\big((f|_V)^{<\gamma>}(x^{(1)}, \ldots, x^{(i-1)}, x_0^{(i)}, \ldots, x_{k-1}^{(i)}, x_{k+1}^{(i)}, \ldots, x_{\gamma_i}^{(i)}, x^{(i+1)}, \ldots, x^{(n)})$$
$$-(f|_V)^{<\gamma>}(x^{(1)}, \ldots, x^{(i-1)}, x_0^{(i)}, \ldots, x_{j-1}^{(i)}, x_{j+1}^{(i)}, \ldots, x_{\gamma_i}^{(i)}, x^{(i+1)}, \ldots, x^{(n)})\big), \quad (10)$$

[5] See [5, Lemma 4.9] for the paradigmatic case of C^r-functions on open sets, [20, Remark II.4. (i) and Prop. II.5] for C^r-functions on locally cartesian sets in Nagel's setting.

where we abbreviated $\gamma := (\gamma_1, \ldots, \gamma_n) := \beta'_i$. Let $D_\beta := \bigcup_{i,j,k} D_{\beta,i,j,k}$. Then $V^{<\beta>} \setminus D_\beta$ is the set of all $x(t) \in \mathbb{K}^{n+|\beta|}$ of the form $x(t) = (x^{(1)}, \ldots, x^{(n)})$ with $x^{(i)} = (t_i, \ldots, t_i)$ for $i \in \{1, \ldots, n\}$, with $t = (t_1, \ldots, t_n) \in V$. Let $W_t \subseteq V$ be an open cartesian neighbourhood of t such that $f|_{W_t}$ is C^α (cf. 2.3 and Remark 2.23). Since W_t is open in V, its i-th component $W_{t,i} \subseteq \mathbb{K}$ is open in the i-th component V_i of the cartesian set V, for $i \in \{1, \ldots, n\}$. Now

$$V^{<\beta>} = D_\beta \cup \bigcup_{t \in V} (W_t)^{<\beta>},$$

where D_β and each $W_t^{<\beta>} = W_{t,1}^{1+\beta_1} \times \cdots \times W_{t,n}^{1+\beta_n}$ is open in $V^{<\beta>} = V_1^{1+\beta_1} \times \cdots \times V_n^{1+\beta_n}$. We define $(f|_V)^{<\beta>} \colon V^{<\beta>} \to E$ via

$$(f|_V)^{<\beta>}(x) := \begin{cases} f_{\beta,i,j,k}(x) & \text{if } x \in D_{\beta,i,j,k}\,; \\ (f|_{W_t})^{<\beta>}(x) & \text{if } x \in W_t^{<\beta>}. \end{cases} \tag{11}$$

To see that this mapping is well defined, let g and h be two of the maps used in the piecewise definition. Let G and H be the domain of h and g, respectively. Then G and H are open in $V^{<\alpha>}$, hence also $G \cap H$. Thus $V^{>\beta<} \cap (G \cap H)$ is dense in $G \cap H$. Since g and h are continuous and coincide on $V^{>\beta<} \cap (G \cap H)$ (where they coincide with $(f|_V)^{>\beta<}$, using Lemmas 2.13 and 2.14), it follows that $g|_{G \cap H} = h|_{G \cap H}$.

Next, because each $g \colon G \to E$ coincides on $G \cap V^{>\beta<}$ with $(f|_V)^{>\beta<}$, we see that $(f|_V)^{<\beta>}|_{V^{>\beta<}} = (f|_V)^{>\beta<}$. This completes the proof. $\square$

Remark 2.25 Assume that $U \subseteq \mathbb{K}^n$ is open. Then U is locally cartesian and $U^{>\beta<}$ is dense in $U^{<\beta>}$ for each $\beta \in \mathbb{N}_0^n$ (as is easy to see). Moreover, if $f \colon U \to E$ is C^α, then $f^{>\beta<}$ admits a continuous extension $f^{<\beta>} \colon U^{<\beta>} \to E$ to all of $U^{<\beta>}$, for each $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$ (let $V := U$ and construct continuous maps $f^{<\beta>} = (f|_V)^{<\beta>}$ verbatim as in the proof of Lemma 2.24).

We record a simple version of the Chain Rule.

Lemma 2.26 *Let $f \colon \mathbb{K}^n \supseteq U \to E$ be a C^α-map and $\lambda \colon E \to F$ be a continuous linear map between topological $\mathbb{K}$-vector spaces. Then $\lambda \circ f$ is C^α, and*

$$(\lambda \circ f|_V)^{<\beta>} = \lambda \circ (f|_V)^{<\beta>} \tag{12}$$

holds for each $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$ and each open cartesian subset $V \subseteq U$.

Proof. The linearity of λ implies that $(\lambda \circ f|_V)^{>\beta<} = \lambda \circ (f|_V)^{>\beta<}$. Because the right-hand side admits the continuous extension $\lambda \circ (f|_V)^{<\beta>}$, the assertions follow. $\square$

The next three lemmas are analogues of [5, Lemmas 10.1–10.3].

Lemma 2.27 *Let $(E_j)_{j\in J}$ be a family of topological $\mathbb{K}$-vector spaces and $E := \prod_{j\in J} E_j$ be the direct product. Let $\mathrm{pr}_j \colon E \to E_j$ be the canonical projections, $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$ and $U \subseteq \mathbb{K}^n$ be a locally cartesian subset. Then a map $f = (f_j)_{j\in J} \colon U \to E$ is C^α if and only if all of its components f_j are C^α. In this case,*

$$(f|_V)^{<\beta>} = ((f_j|_V)^{<\beta>})_{j\in J} \tag{13}$$

for all open cartesian subsets $V \subseteq U$ and all $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$.

Proof. If f is C^α, then also $f_j = \mathrm{pr}_j \circ f$ is C^α, by Lemma 2.26. Conversely, assume that each f_j is C^α. For V and β as above, we have $(f|_V)^{>\beta<} = ((f_j|_V)^{>\beta<})_{j\in J}$. Because $((f_j|_V)^{<\beta>})_{j\in J}$ is a continuous extension for the right-hand side, we deduce that f is C^α and (13) holds. $\square$

Lemma 2.28 *Let E be a topological $\mathbb{K}$-vector space, $E_0 \subseteq E$ be a closed vector subspace, $U \subseteq \mathbb{K}^n$ be a locally cartesian subset and $f \colon U \to E_0$ be a map. Then f is C^α as a map to E_0 if and only if f is C^α as a map to E.*

Proof. The inclusion map $\iota \colon E_0 \to E$ is continuous and linear. Hence, if $f \colon U \to E_0$ is C^α, then also $\iota \circ f \colon U \to E$ is C^α, by Lemma 2.26. Conversely, assume that $\iota \circ f$ is C^α. Let $V \subseteq U$ be an open cartesian set and $\beta \in \mathbb{N}_0^n$ with $\beta \leq \alpha$. Let $x \in V^{<\beta>}$. Since $V^{>\beta<}$ is dense in $V^{<\beta>}$, there is a net $(x_a)_{a\in A}$ in $V^{>\beta<}$ such that $x_a \to x$. Then $(\iota \circ f|_V)^{<\beta>}(x) = \lim (\iota \circ f|_V)^{<\beta>}(x_a)$, where $(\iota \circ f|_V)^{<\beta>}(x_a) = (\iota \circ f|_V)^{>\beta<}(x_a) = (f|_V)^{>\beta<}(x_a) \in E_0$ for each $a \in A$. Since E_0 is closed, we deduce that $(\iota \circ f|_V)^{<\beta>}(x) \in E_0$. Thus $(\iota \circ f|_V)^{<\beta>} \colon V^{<\beta>} \to E_0$ is a continuous extension to $(f|_V)^{>\beta<}$ and we deduce that f is C^α. $\square$

If $\mathbb{K}$ is metrizable, then it suffices to assume that E_0 is sequentially closed in Lemma 2.28 (because the net $(x_a)_{a\in A}$ can be replaced with a sequence then).

Lemma 2.29 *Let $(J, \leq)$ be a directed set, $\mathcal{S} := ((E_j)_{j\in J}, (\phi_{i,j})_{i\leq j})$ be a projective system of topological $\mathbb{K}$-vector spaces E_j, with continuous linear maps $\phi_{i,j}\colon E_j \to E_i$ satisfying $\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k}$ for $i \leq j \leq k$ in J, and $\phi_{i,i} = \mathrm{id}_{E_i}$. Let $(E, (\phi_i)_{i\in I})$ be a projective limit of $\mathcal{S}$, with the continuous linear maps $\phi_j\colon E \to E_j$ such that $\phi_{j,k} \circ \phi_k = \phi_j$. Then a map $f\colon \mathbb{K}^n \supseteq U \to E$ is C^α if and only if $\phi_j \circ f$ is C^α for each $j \in J$.*

Proof. After applying an isomorphism of topological vector spaces, we may assume that E is realized in the usual form as a closed vector subspace of the direct product $\prod_{j\in J} E_j$, and $\phi_j = \mathrm{pr}_j|_E$. The assertion now follows from Lemmas 2.27 and 2.28. □

3 The compact-open C^α-topology

Throughout this section, $\mathbb{K}$ is a topological field, E a topological $\mathbb{K}$-vector space, $n \in \mathbb{N}$, $U \subseteq \mathbb{K}^n$ a locally cartesian subset and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$.

Definition 3.1 We endow that space $C^\alpha(U, E)$ of all E-valued C^α-maps on U with the initial topology $\mathcal{O}$ with respect to the mappings

$$\Delta_{\beta,V}\colon C^\alpha(U,E) \to C(V^{<\beta>}, E), \quad f \mapsto (f|_V)^{<\beta>}, \tag{14}$$

for all $\beta \in \mathbb{N}_0^n$ with $\beta \leq \alpha$ and open cartesian subsets $V \subseteq U$; the spaces $C(V^{<\beta>}, E)$ on the right-hand side are endowed with the compact-open topology. We call $\mathcal{O}$ the *compact-open C^α-topology.*

Remark 3.2 (a) Since each $C(V^{<\beta>}, E)$ is a topological vector space and the mappings in (14) are linear (and separate points), also $C^\alpha(U, E)$ is a topological vector space.

(b) If $\alpha = 0$, our definition yields the compact-open topology on $C^0(U, E)$, by Lemma B.11.

(c) If $\alpha' \in (\mathbb{N}_0 \cup \{\infty\})^n$ with $\alpha' \leq \alpha$, then the inclusion mapping $\iota\colon C^\alpha(U, E) \to C^{\alpha'}(U, E)$ is continuous.

[For $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha'$, let the mapping $\Delta_{\beta,V}$ be as in (14) and $\Delta'_{\beta,V}\colon C^{\alpha'}(U, E) \to C(V^{<\beta>}, E)$ be the analogous map. Then $\Delta'_{\beta,V} \circ \iota = \Delta_{\beta,V}$ is continuous, from which the assertion follows.]

(d) In particular, the compact-open C^α-topology on $C^\alpha(U,E)$ is always finer than the compact-open topology (induced from $C(U,E)$).

Lemma 3.3 *Let $(V_a)_{a\in A}$ be a cover of U by open cartesian subsets $V_a \subseteq U$. Then the compact-open C^α-topology on $C^\alpha(U,E)$ is the initial topology $\mathcal{T}$ with respect to the maps Δ_{β,V_a}, for $\beta\in\mathbb{N}_0^n$ such that $\beta\leq\alpha$ and $a\in A$.*

Proof. By definition, the compact-open C^α-topology $\mathcal{O}$ makes each of the maps Δ_{β,V_a} continuous. Hence $\mathcal{T}\subseteq\mathcal{O}$. To see that $\mathcal{O}\subseteq\mathcal{T}$, we have to show that $\Delta_{\beta,V}$ is continuous as a map on $(C^\alpha(U,E),\mathcal{T})$, for each $\beta\in\mathbb{N}_0^n$ such that $\beta\leq\alpha$ and cartesian open subset $V\subseteq U$.

Given β and V, pick open cartesian subsets $W_t\subseteq V$ for $t\in V$ such that $W_t\subseteq V_{a_t}$ for some $a_t\in A$. Define I_β and open sets $(W_t)^{<\beta>}$ and $D_{\beta,i,j,k}$ as in the proof of Lemma 2.24, for $i\in I_\beta$ and $j<k$ in $\{0,1,\ldots,\beta_i\}$. Let

$$\rho_t\colon C(V^{<\beta>},E)\to C((W_t)^{<\beta>},E),\quad \sigma_t\colon C((V_{a_t})^{<\beta>},E)\to C((W_t)^{<\beta>},E)$$

and $\rho_{i,j,k}\colon C(V^{<\beta>},E)\to C(D_{\beta,i,j,k},E)$ be the respective restriction maps. Since $V^{<\beta>}$ is the union of the open sets $(W_t)^{<\beta>}$ and $D_{\beta,i,j,k}$, we deduce with Lemma B.11 that $\mathcal{T}$ will make $\Delta_{\beta,V}$ continuous if it makes all of the maps $\rho_t\circ\Delta_{\beta,V}$ and $\rho_{i,j,k}\circ\Delta_{\beta,V}$ continuous. We first note that $\mathcal{T}$ makes $\rho_t\circ\Delta_{\beta,V}=\sigma_t\circ\Delta_{\beta,V_{a_t}}$ continuous. If $\beta=0$, this implies the continuity of the restriction map $\Delta_{0,V}$ (since $V=V^{<0>}$ is covered by the sets $W_t=(W_t)^{<0>}$). Now assume that $|\beta|\geq 1$ and assume, by induction, that $\mathcal{T}$ makes $\Delta_{\gamma,V}$ continuous for all $\gamma\in\mathbb{N}_0^n$ such that $\gamma\leq\alpha$ and $|\gamma|<|\beta|$. Fix $i\in I_\beta$ and $j<k$ in $\{0,1,\ldots,\beta_i\}$. Abbreviate $\gamma:=\beta-e_i$. The map

$$h\colon D_{\beta,i,j,k}\to\mathbb{K},\quad x=(x^{(1)},\ldots,x^{(n)})\mapsto\frac{1}{x_j^{(i)}-x_k^{(i)}}$$

is continuous (where $D_{\beta,i,j,k}$ is as in the proof of Lemma 2.24). Hence also the multiplication operator

$$m_h\colon C(D_{\beta,i,j,k},E)\to C(D_{\beta,i,j,k},E),\quad g\mapsto h\cdot g$$

is continuous, by Lemma B.21 (f). Next, let $g_1,g_2\colon D_{\beta,i,j,k}\to V^{<\gamma>}$ be the mappings taking $x=(x^{(1)},\ldots,x^{(n)})$ to

$$x^{(1)},\ldots,x^{(i-1)},x_0^{(i)},\ldots,x_{k-1}^{(i)},x_{k+1}^{(i)},\ldots,x_{\gamma_i}^{(i)},x^{(i+1)},\ldots,x^{(n)})$$

and $x^{(1)}, \ldots, x^{(i-1)}, x_0^{(i)}, \ldots, x_{j-1}^{(i)}, x_{j+1}^{(i)}, \ldots, x_{\gamma_i}^{(i)}, x^{(i+1)}, \ldots, x^{(n)})$, respectively. Then g_1 and g_2 are continuous, entailing that also the pullbacks $C(g_1, E)$ and $C(g_2, E)$ are continuous as mappings $C(V^{<\gamma>}, E) \to C(D_{\beta,i,j,k}, E)$. By (10) and (11), we have

$$\rho_{i,j,k}(\Delta_{\beta,V}(f)) = (f|_V)^{<\beta>}|_{D_{\beta,i,j,k}} = f_{\beta,i,j,k} = h\cdot((f|_V)^{<\gamma>} \circ g_1 - (f|_V)^{<\gamma>} \circ g_2).$$

Hence $\rho_{i,j,k} \circ \Delta_{\beta,V} = m_h \circ (C(g_1, E) \circ \Delta_{\gamma,V} - C(g_2, E) \circ \Delta_{\gamma,V})$. The right-hand side is composed of continuous maps and hence continuous. Thus $\rho_{i,j,k} \circ \Delta_{\beta,V}$ is continuous on $(C^\alpha(U, E), \mathcal{T})$. □

Lemma 3.4 *If $U \subseteq \mathbb{K}^n$ is open or cartesian, then the compact-open C^α-topology on $C^\alpha(U, E)$ is the initial topology with respect to the maps*

$$\Delta_\beta \colon C^\alpha(U, E) \to C(U^{<\beta>}, E), \quad f \mapsto f^{<\beta>},$$

for $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$.

Again, we use the compact-open topology on $C(U^{<\beta>}, E)$ here.

Proof. The preceding proof can be repeated verbatim with $V := U$. □

Lemma 3.5 *Let $S \subseteq \mathbb{K}^n$ be a locally cartesian subset such that $S \subseteq U$. Then $f|_S$ is C^α for each C^α-map $f \colon U \to E$. Moreover, the "restriction map"*

$$\rho \colon C^\alpha(U, E) \to C^\alpha(S, E), \quad f \mapsto f|_S$$

is continuous and linear.

Proof. For $x \in S$, let $V_x \subseteq U$ and $W_x \subseteq S$ be open cartesian subsets containing x. By 2.3, we may assume that $V_x = U \cap Q$ and $W_x = S \cap P$ with open cartesian subsets Q, P of $\mathbb{K}^n$. After replacing P and Q with their open cartesian subset $P \cap Q$ (which is possible by the proof of 2.3), we may assume that $V_x = U \cap Q$ and $W_x = S \cap Q$, whence $W_x = S \cap V_x \subseteq V_x$. Because the sets W_x form a cover of S for $x \in S$, and $(f|_{V_x})^{<\beta>}|_{(W_x)^{<\beta>}}$ is a continuous extension of $(f|_{W_x})^{>\beta<}$ for each $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$, we deduce with Lemma 2.24 that $f|_S$ is C^α, with

$$(f|_{W_x})^{<\beta>} = (f|_{V_x})^{<\beta>}|_{(W_x)^{<\beta>}}. \tag{15}$$

Clearly ρ is linear. Define $\Delta_{\beta,V_x}\colon C^\alpha(U,E)\to C((V_x)^{<\beta>},E)$, $f\mapsto (f|_{V_x})^{<\beta>}$ and $\Delta^S_{\beta,W_x}\colon C^\alpha(S,E)\to C((W_x)^{<\beta>},E)$, $f\mapsto (f|_{W_x})^{<\beta>}$. Let $\rho_{\beta,x}\colon C((V_x)^{<\beta>},E)\to C((W_x)^{<\beta>},E)$ be the restriction map. Then (15) can be rewritten as

$$\Delta^S_{\beta,W_x}\circ\rho=\rho_{\beta,x}\circ\Delta_{\beta,V_x}\,. \tag{16}$$

The right-hand side of (16) is continuous by Remark B.10 and the continuity of the maps Δ_{β,V_x}. Hence also the left-hand side is continuous and therefore ρ is continuous, using that the topology on $C^\alpha(S,E)$ is initial with respect to the maps Δ^S_{β,W_x}, by Lemma 3.3. □

Lemma 3.6 *Let $(U_a)_{a\in A}$ be a cover of U by open subsets $U_a\subseteq U$. Then the compact-open C^α-topology on $C^\alpha(U,E)$ is the initial topology $\mathcal{T}$ with respect to the restriction maps*

$$\rho_a\colon C^\alpha(U,E)\to C^\alpha(U_a,E),\quad f\mapsto f|_{U_a}\qquad \textit{for } a\in A.$$

Proof. For $a\in A$, let $\mathcal{V}_a$ be the set of all cartesian open subsets $V\subseteq U_a$. For $V\in\mathcal{V}_a$ and $\beta\in\mathbb{N}_0^n$ such that $\beta\leq\alpha$, let $\Delta_{\beta,V}\colon C^\alpha(U,E)\to C(V^{<\beta>},E)$ and $\Delta^a_{\beta,V}\colon C^\alpha(U_a,E)\to C(V^{<\beta>},E)$ be defined via $f\mapsto (f|_V)^{<\beta>}$. Because the topology on $C^\alpha(U_a,E)$ is initial with respect to the maps $\Delta^a_{\beta,V}$ with β and V as before, the "transitivity of initial topologies" implies that $\mathcal{T}$ is the initial topology with respect to the mappings $\Delta^a_{\beta,V}\circ\rho_a=\Delta_{\beta,V}$. As $\bigcup_{a\in A}\mathcal{V}_a$ is a cover of U by cartesian open subsets of U, the latter topology coincides with the compact-open C^α-topology on $C^\alpha(U,E)$, by Lemma 3.3. □

Lemma 3.7 *For each cover $\mathcal{V}$ of U by open cartesian sets $V\subseteq U$, the map*

$$\Delta:=(\Delta_{\beta,V})\colon C^\alpha(U,E)\to\prod_{\beta,V}C(V^{<\beta>},E),\quad f\mapsto((f|_V)^{<\beta>})_{\beta,V} \tag{17}$$

is linear and a topological embedding with closed image (where the product is taken over all $V\in\mathcal{V}$ and all $\beta\in\mathbb{N}_0^n$ such that $\beta\leq\alpha$).

Proof. The linearity is clear. As $\Delta_{0,V}$ is the restriction map $f\mapsto f|_V$ and $\mathcal{V}$ is a cover of U, the map Δ is injective. Combining this with Lemma 3.3, we find that Δ is a topological embedding. To see that the image is closed, let $(f_a)_{a\in A}$ be a net in $C^\alpha(U,E)$ such that $\Delta(f_a)\to(g_{\beta,V})_{\beta,V}$ with functions $g_{\beta,V}\in$

$C(V^\beta, E)$. If $x \in U$, there is $V \in \mathcal{V}$ such that $x \in V$. Then $f(x) := \lim f_a(x)$ exists, because $f_a(x) = \Delta_{0,V}(f_a)(x) \to g_{0,V}(x)$. By the preceding, $f|_V = g_{0,V}$, whence f is continuous. It is clear from (4) that $((f_a)|_V)^{>\beta<}$ converges pointwise to $(f|_V)^{>\beta<}$. On the other hand, $((f_a)|_V)^{>\beta<} = \Delta_{\beta,V}(f_a)|_{V^{>\beta<}}$ converges pointwise to $g_{\beta,V}|_{V^{>\beta<}}$. Hence $(f|_V)^{>\beta<} = g_{\beta,V}|_{V^{>\beta<}}$. Since $g_{\beta,V}$ provides a continuous extension for the latter function, we see with Lemma 2.24 that f is C^α, with $(f|_V)^{<\beta>} = g_{\beta,V}$ for all $V \in \mathcal{V}$ and $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$. □

Lemma 3.8 *Let B_α be the set of all $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$. The spaces $C^\beta(U, E)$ form a projective system of topological $\mathbb{K}$-vector spaces for $\beta \in B_\alpha$, together with the inclusion maps $\phi_{\gamma,\beta}\colon C^\beta(U, E) \to C^\gamma(U, E)$ for $\gamma \leq \beta$ in B_α. Moreover, $C^\alpha(U, E)$ is the projective limit of this system, together with the inclusion maps $\phi_\beta\colon C^\alpha(U, E) \to C^\beta(U, E)$ for $\beta \in B_\alpha$.*

Proof. If $(f_\beta)_{\beta \in B_\alpha}$ is an element of

$$\varprojlim C^\beta(U, E) = \{(f_\beta)_{\beta \in B_\alpha} \in \prod_{\beta \in B_\alpha} C^\beta(U, E) \colon (\beta \geq \gamma) \Rightarrow f_\gamma = \phi_{\gamma,\beta}(f_\beta)\},$$

then $\phi_{\gamma,\beta}(f_\beta) = f_\gamma$ and thus $f_\beta = f_\gamma$, whenever $\beta \geq \gamma$. Hence $f_0 = f_\beta$ for all $\beta \in B_\alpha$, and thus $f_0 \in C^\alpha(U, E)$. As a consequence, the map

$$\Psi\colon C^\alpha(U, E) = \bigcap_{\beta \in B_\alpha} C^\beta(U, E) \to \varprojlim C^\beta(U, E), \quad f \mapsto (f)_{\beta \in B_\alpha}$$

is an isomorphism of vector spaces. Let $\Delta_{\beta,V}\colon C^\alpha(U, E) \to C(V^{<\beta>}, E)$ be as in (14), $\Delta^\beta_{\gamma,V}\colon C^\beta(U, E) \to C(V^{<\gamma>}, E)$ for $\gamma \leq \beta$ be the analogous map and $\pi_\beta\colon \varprojlim C^\beta(U, E) \to C^\beta(U, E)$ be the projection onto the β-component. Using the transitivity of initial topologies twice, we see: The initial topology on $C^\alpha(U, E)$ with respect to Ψ coincides with the initial topology with respect to the maps $\pi_\beta \circ \Psi$; and this topology is initial with respect to the maps $\Delta^\beta_{\gamma,V} \circ \pi_\beta \circ \Psi = \Delta_{\gamma,V}$. It therefore coincides with the compact-open C^α-topology on $C^\alpha(U, E)$, and hence Ψ is a homeomorphism. □

Recall that a topological space X is called *hemicompact* if there exists a sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets of X, with $\bigcup_{n \in \mathbb{N}} K_n = X$, such that each compact subset of X is contained in some K_n. For example, every σ-compact locally compact space is hemicompact.

Also, we recall: For each point x in a locally compact topological space X and open neighbourhood P of x, there exists a σ-compact open neighbourhood Q of x in X such that $Q \subseteq P$.[6]

Lemma 3.9 *If $U \subseteq \mathbb{K}^n$ is a locally cartesian subset which is locally compact and σ-compact, then U admits a countable cover by cartesian open subsets $V \subseteq U$ which are σ-compact.*

Proof. It suffices to show that each $x \in U$ has an open cartesian neighbourhood $V_x \subseteq U$ which is σ-compact. In fact, writing $U = \bigcup_{n\in\mathbb{N}} C_n$ with compact sets C_n, each C_n will be covered by V_x with x in a finite subset $\Phi_n \subseteq C_n$. Then the V_x for x in the countable set $\bigcup_{n\in\mathbb{N}} \Phi_n$ provide the desired countable cover for U.

Fix $x = (x_1, \ldots, x_n) \in U$. To construct V_x, start with a cartesian open neighbourhood $W = W_1 \times \cdots \times W_n \subseteq U$ of x. Since U is locally compact, there exists a compact neighbourhood $K \subseteq W$ of x. Let $\mathrm{pr}_j \colon \mathbb{K}^n \to \mathbb{K}$ be the projection onto the j-th component, for $j \in \{1, \ldots, n\}$. Then $K_j := \mathrm{pr}_j(K)$ is a compact subset of W_j and thus $K_1 \times \cdots \times K_n \subseteq W$. After replacing K with a larger set, we may therefore assume that $K = K_1 \times \cdots \times K_n$. Let K^0 be the interior of K relative W. Then $K^0 = K_1^0 \times \cdots \times K_n^0$, where K_j^0 is the interior of K_j relative W_j. Because K_j is compact and K_j^0 an open neighbourhood of x_j in K_j, there exists a σ-compact, open neighbourhood Q_j of x_j in K_j such that $Q_j \subseteq K_j^0$. Then Q_j is open in K_j^0 and hence also open in W_j, whence Q_j does not have isolated points and $V_x := Q_1 \times \cdots \times Q_n$ is open in W and hence in U. Since each Q_j is σ-compact, so is the finite direct product V_x. □

Lemma 3.10 *If $V \subseteq \mathbb{K}^n$ is a cartesian subset which is locally compact and σ-compact, then $V^{<\beta>}$ is locally compact and σ-compact, for each $\beta \in \mathbb{N}_0^n$.*

Proof. Let $\mathrm{pr}_j \colon \mathbb{K}^n \to \mathbb{K}$ be the projection onto the j-th component, for $j \in \{1, \ldots, n\}$. We may assume $V \neq \emptyset$. Write $V = V_1 \times \cdots \times V_n$ with $V_j \subseteq \mathbb{K}$. As V is locally compact, each V_j is locally compact. Further, $V_j = \mathrm{pr}_j(V)$ is

[6]If $L \subseteq P$ is compact, then each $y \in L$ has a compact neighbourhood L_y in P. By compactness of L, we have $L \subseteq \bigcup_{y\in\Phi} L_y =: L'$ for some finite subset $\Phi \subseteq L$. Then L is in the interior of L' and $L' \subseteq P$. Starting with $L_1 := \{x\}$, this construction yields compact subsets $L_{n+1} := (L_n)'$ of P for $n \in \mathbb{N}$ such that L_n is in the interior L_{n+1}^0. Thus $Q = \bigcup_{n\in\mathbb{N}} L_n = \bigcup_{n\in\mathbb{N}} L_n^0$ has the desired properties.

σ-compact. Hence also each $V^{<\beta>}$ is locally compact and σ-compact, being a finite direct product of copies of the factors $V_1, \ldots, V_n$ (see (3)). $\square$

Lemma 3.11 *The space $C^\alpha(U, E)$ has the following properties:*

(a) *If $\mathbb{K}$ is metrizable and E is complete, then $C^\alpha(U, E)$ is complete.*

(b) *If $\mathbb{K}$ is metrizable and E is sequentially complete, then $C^\alpha(U, E)$ is sequentially complete.*

(c) *If U is σ-compact and locally compact and E is metrizable, then $C^\alpha(U, E)$ is metrizable.*

(d) *If $\mathbb{K}$ is an ultrametric field and E is locally convex, then $C^\alpha(U, E)$ is locally convex.*

(e) *If $\mathbb{K}$ is an ultrametric field, U is compact, E an ultrametric normed space and $\alpha \in \mathbb{N}_0^n$, then $C^\alpha(U, E)$ is an ultrametric normed space.*

(f) *If $\mathbb{K}$ is an ultrametric field, U is compact, E an ultrametric Banach space and $\alpha \in \mathbb{N}_0^n$, then $C^\alpha(U, E)$ is an ultrametric Banach space.*

(g) *Assume that there exists a cover $\mathcal{V}$ of U by cartesian open subsets $V \subseteq U$ such that $V^{<\beta>}$ is a k-space for each $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$. Then $C^\alpha(U, E)$ is complete if E is complete, and $C^\alpha(U, E)$ is sequentially complete if E is sequentially complete.*

(h) *Let E be metrizable and assume that there exists a countable cover $\mathcal{V}$ of U by cartesian open subsets $V \subseteq U$ such that $V^{<\beta>}$ is hemicompact for each $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$. Then $C^\alpha(U, E)$ is metrizable.*

Proof. (a) and (b) follow from (g), because every metrizable topological space is a k-space.

(c) By Lemma 3.9, U admits a countable cover $\mathcal{V}$ by cartesian open subsets $V \subseteq U$ which are locally compact and σ-compact. For such V and $\beta \in \mathbb{N}_0^n$ with $\beta \leq \alpha$, the set $V^{<\beta>}$ is locally compact and σ-compact (by Lemma 3.10) and hence hemicompact. Hence (c) follows from (h).

(d) By Lemma 3.7, the topological vector space $C^\alpha(U, E)$ is isomorphic to a vector subspaces of a direct product of the spaces of the form $C(V^{<\beta>}, E)$. Each of these is locally convex by Lemma B.21 (g), and hence also $C^\alpha(U, E)$.

(e) Assume first that the compact set U is cartesian, $U_1 \times \cdots \times U_n = U$ with compact subsets $U_1, \ldots, U_n \subseteq \mathbb{K}$. We can then take $\mathcal{V} = \{U\}$ in Lemma 3.7. Hence $C^\alpha(U, E)$ is isomorphic to a vector subspaces of a finite direct product of spaces of the form $C(U^{<\beta>}, E)$. Each $C(U^{<\beta>}, E)$ is an ultrametric normed space by Lemma B.21 (h), using that $U^{<\beta>}$ is of the form (3) and hence compact. Hence also $C(U^{<\beta>}, E)$ is an ultrametric normed space.

We now return to the general case, assuming only that the compact set U is locally cartesian. Fix $x \in U$. Then x has a cartesian open neighbourhood $W = W_1 \times \cdots \times W_n \subseteq U$ and the latter contains a compact neighbourhood K of x of the form $K = K_1 \times \cdots \times K_n$ (see proof of Lemma 3.9). We let $V_x := K^0$ be the interior of K relative W. Then $V_x = K_1^0 \times \cdots \times K_n^0$, where K_j^0 is the interior of K_j relative W_j. Since W_j does not have isolated points and K_j^0 is open in W_j, also K_j^0 does not have isolated points, and hence also its closure $\overline{K_j^0}$ does not have isolated points. Hence $L_x := \overline{V_x} = \overline{K_1^0} \times \cdots \times \overline{K_n^0}$ is a compact cartesian subset of $\mathbb{K}^n$. Since V_x is a neighbourhood of x in U, by compactness of U there is a finite subset $\Phi \subseteq U$ such that $U = \bigcup_{x \in \Phi} V_x$. For each $x \in \Phi$, the restriction maps

$$\rho_x \colon C^\alpha(U, E) \to C^\alpha(L_x, E) \quad \text{and} \quad \sigma_x \colon C^\alpha(L_x, E) \to C^\alpha(V_x, E)$$

are continuous linear, by Lemma 3.5. Hence the mappings

$$\rho := (\rho_x)_{x \in \Phi} \colon C^\alpha(U, E) \to \prod_{x \in \Phi} C^\alpha(L_x, E)$$

and

$$\sigma := \prod_{x \in \Phi} \sigma_x \colon \prod_{x \in \Phi} C^\alpha(L_x, E) \to \prod_{x \in \Phi} C^\alpha(V_x, E), \quad (f_x)_{x \in \Phi} \mapsto (\sigma_x(f_x))_{x \in \Phi}$$

are continuous linear. Because $\sigma \circ \rho$ is a topological embedding (by Lemma 3.7), we deduce that also ρ is a topological embedding. Since each of the spaces $C^\alpha(L_x, E)$ is an ultrametric normed space, so is their finite direct product $\prod_{x \in \Phi} C^\alpha(L_x, E)$ and hence also $C^\alpha(U, E)$.

(f) follows from (a) and (e).

(g) By Lemma 3.7, the topological vector space $C^\alpha(U, E)$ is isomorphic to a closed vector subspace of the product of the spaces $C(V^{<\beta>}, E)$ with β and V as described in (g). Each of the latter is complete (resp., sequentially complete), by Lemma B.21 (d) and (e). Hence so is $C^\alpha(U, E)$.

(h) Let B_α be the set of all $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$. Since B_α and $\mathcal{V}$ are countable, also $B_\alpha \times \mathcal{V}$ is countable. By Lemma B.21 (c), each of the spaces $C(V^{<\beta>}, E)$ is metrizable. Hence also the countable direct product $\prod_{(\beta,V)\in B_\alpha \times \mathcal{V}} C(V^{<\beta>}, E)$ is metrizable and hence also $C^\alpha(U, E)$, being homeomorphic to a subspace of the latter product by Lemma 3.7. $\square$

Lemma 3.12 *If $\lambda\colon E \to F$ is a continuous linear mapping between topological $\mathbb{K}$-vector spaces, then the map*

$$C^\alpha(U, \lambda)\colon C^\alpha(U, E) \to C^\alpha(U, F), \quad f \mapsto \lambda \circ f$$

is continuous and linear. If λ is linear and a topological embedding, then also $C^\alpha(U, \lambda)$ is linear and a topological embedding.

Proof. Let $\Delta_{\beta,V}\colon C^\alpha(U, E) \to C(V^{<\beta>}, E)$ be as in Definition 3.1 and $\Delta^F_{\beta,V}\colon C^\alpha(U, F) \to C(V^{<\beta>}, F)$ be the analogous map. By Lemma 2.26,

$$\Delta^F_{\beta,V} \circ C^\alpha(U, \lambda) = C(V^{<\beta>}, \lambda) \circ \Delta_{\beta,V}. \tag{18}$$

Because the right-hand side of (18) is continuous, also each of the maps $\Delta^F_{\beta,V} \circ C^\alpha(U, \lambda)$ is continuous and hence also $C^\alpha(U, \lambda)$ (as the topology on $C^\alpha(U, F)$ is initial with respect to the mappings $\Delta^F_{\beta,V}$).

If λ is an embedding, then the topology on E is initial with respect to λ. Since $C^\alpha(U, \lambda)$ is injective, it will be an embedding if we can show that the topology on $C^\alpha(U, E)$ is initial with respect to $C^\alpha(U, \lambda)$. This is a special case of the next lemma. $\square$

Lemma 3.13 *If the topology on E is initial with respect to a family $(\lambda_j)_{j\in J}$ of linear maps $\lambda_j\colon E \to E_j$ to topological $\mathbb{K}$-vector spaces E_j, then the compact-open C^α-topology on $C^\alpha(U, E)$ is the initial topology with respect to the linear maps $C^\alpha(U, \lambda_j)\colon C^\alpha(U, E) \to C^\alpha(U, E_j)$, for $j \in J$.*

Proof. Let $\mathcal{O}$ be the compact-open C^α-topology on $C^\alpha(U, E)$ and $\mathcal{T}$ be the initial topology described in the lemma.

Step 1. By Lemma B.4, the topology on $C(V^{<\beta>}, E)$ is initial with respect to the maps $C(V^{<\beta>}, \lambda)\colon C(V^{<\beta>}, E) \to C(V^{<\beta>}, E_j)$. Thus, by transitivity of initial topologies, $\mathcal{O}$ is initial with respect to the maps $C(V^{<\beta>}, \lambda_j) \circ \Delta_{\beta,V}$

with $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$, open cartesian subsets $V \subseteq U$ and $\Delta_{\beta,V}$ as in Definition 3.1.

Step 2. Let $\Delta^{E_j}_{\beta,V}\colon C^\alpha(U,E_j) \to C(V^{<\beta>},E_j)$ be the analogous maps. The topology on $C^\alpha(U,E_j)$ is initial with respect to the maps $\Delta^{E_j}_{\beta,V}$. Hence, by transitivity of initial topologies, the topology $\mathcal{T}$ is initial with respect to the maps $\Delta^{E_j}_{\beta,V} \circ C^\alpha(U,\lambda_j)$. But $\Delta^{E_j}_{\beta,V} \circ C^\alpha(U,\lambda_j) = C(V^{<\beta>},\lambda_j) \circ \Delta_{\beta,V}$, by Lemma 2.26. Comparing with Step 1, we see that $\mathcal{O} = \mathcal{T}$. $\square$

Lemma 3.14 *If $E_0 \subseteq E$ is a vector subspace, then the compact-open C^α-topology on $C^\alpha(U,E_0)$ coincides with the topology induced by $C^\alpha(U,E)$.*

Proof. Apply Lemma 3.12 to the inclusion map $\lambda\colon E_0 \to E$. $\square$

Lemma 3.15 *If $E = \prod_{j\in J} E_j$ as a topological $\mathbb{K}$-vector space, with the canonical projections $\mathrm{pr}_j\colon E \to E_j$, then the mapping*

$$\Psi := (C^\alpha(U,\mathrm{pr}_j))_{j\in J}\colon C^\alpha(U,E) \to \prod_{j\in J} C^\alpha(U,E_j)$$

is an isomorphism of topological vector spaces.

Proof. Lemma 2.27 entails that Ψ is bijective and hence an isomorphism of vector spaces. Because the topology on E is initial with respect to the family $(\mathrm{pr}_j)_{j\in J}$, Lemma 3.13 shows that the topology on $C^\alpha(U,E)$ is initial with respect to the maps $C^\alpha(U,\mathrm{pr}_j)$. Hence Ψ is a topological embedding and hence Ψ is an isomorphism of topological vector spaces. $\square$

Lemma 3.16 *Let $E = \lim\limits_{\longleftarrow} E_j$ be as in Lemma 2.29, with the bonding maps $\phi_{i,j}\colon E_j \to E_i$ and the limit maps $\phi_j\colon E \to E_j$. Then the spaces $C^\alpha(U,E_j)$ form a projective system with bonding maps $C^\alpha(U,\phi_{i,j})$, and*

$$C^\alpha(U,E) = \lim_{\longleftarrow} C^\alpha(U,E_j),$$

with the limit maps $C^\alpha(U,\phi_j)$.

Proof. Let $P := \prod_{j\in J} E_j$ and $\mathrm{pr}_j\colon P \to E_j$ be the projection onto the j-th component. It is known that the map

$$\phi := (\phi_j)_{j\in J}\colon E \to P$$

is a topological embedding with image $\phi(E) = \{(x_j)_{j\in J} \in P\colon (\forall i,j \in J)\; i \le j \Rightarrow x_i = \phi_{i,j}(x_j)\}$. Thus $C^\alpha(U,\phi)$ is a topological embedding (see Lemma 3.12) and hence also

$$\Psi \circ C^\alpha(U,\phi)\colon C^\alpha(U,E) \to \prod_{j\in J} C^\alpha(U,E_j),$$

using the isomorphism $\Psi := (C^\alpha(U,\mathrm{pr}_j))_{j\in J}\colon C^\alpha(U,P) \to \prod_{j\in J} C^\alpha(U,E_j)$ of topological vector spaces from Lemma 3.15. The image of $\Psi \circ C^\alpha(U,\phi)$ is contained in the projective limit

$$L := \{(f_j)_{j\in J} \in \prod_{j\in J} C^\alpha(U,E_j)\colon (\forall i,j \in J)\, i \le j \Rightarrow f_i = \phi_{i,j} \circ f_j\}$$

of the spaces $C^\alpha(U,E_j)$. If $(f_j)_{j\in J} \in L$ and $x \in U$, then $\phi_{i,j}(f_j(x)) = f_i(x)$ for all $i \le j$, whence there exists $f(x) \in E$ with $\phi_i(f(x)) = f_i(x)$. By Lemma 2.29, we have $f \in C^\alpha(U,E)$. Now $(\Psi \circ C^\alpha(U,\phi))(f) = (f_j)_{j\in J}$. Hence $\Psi \circ C^\alpha(U,\phi)$ has image L and hence is an isomorphism of topological vector spaces from $C^\alpha(U,E)$ onto L. The assertions follow. □

The following observation will be useful in the proof of Theorem B.

Lemma 3.17 *For each $f \in C^\alpha(U,\mathbb{K})$ and $a \in E$, we have $fa \in C^\alpha(U,E)$, and the following linear map is continuous:*

$$\theta\colon E \to C^\alpha(U,E), \quad a \mapsto fa.$$

Proof. The map $m_a\colon \mathbb{K} \to E$, $x \mapsto xa$ is continuous linear, whence $fa = m_a \circ f \in C^\alpha(U,E)$ (see Lemma 2.26). Let $\Delta_{\beta,V}\colon C^\alpha(U,E) \to C(V^{<\beta>},E)$ be as in Definition 3.1 and define $\Delta^{\mathbb{K}}_{\beta,V}\colon C^\alpha(U,\mathbb{K}) \to C(V^{<\beta>},\mathbb{K})$ analogously. Then

$$\Delta_{\beta,V} \circ \theta\colon E \to C(V^{<\beta>},E)$$

is continuous. In fact, we can write $\Delta_{\beta,V}(\theta(a)) = \Delta_{\beta,V}(m_a \circ f) = m_a \circ \Delta^{\mathbb{K}}_{\beta,V}(f) = \Delta^{\mathbb{K}}_{\beta,V}(f) \cdot c_a$, where the dot denotes the continuous module multiplication $C(V^{<\beta>},\mathbb{K}) \times C(V^{<\beta>},E) \to C(V^{<\beta>},E)$ from Lemma B.21 (f) and the continuous linear map $E \to C(V^{<\beta>},E)$, $a \mapsto c_a$ is as in Lemma B.17. Hence θ is continuous. □

Remark 3.18 In our terminology, [19, Lemma 3.9] asserts that, for each complete ultrametric field $\mathbb{K}$, locally cartesian subset $U \subseteq \mathbb{K}^n$ and compact set $K \subseteq U$ of the form $K = K_1 \times \cdots \times K_n$, there exists a cartesian open subset $V \subseteq U$ with $K \subseteq V$. The next example shows that this assertion is incorrect.[7] Therefore the definition of the topology of "topology of compact cartesian convergence" in [20, pp. 24–25] does not work as stated (instead, one should only consider compact sets which are contained in some cartesian open subset V of the domain of definition).

Example 3.19 Let $U = (V_1 \times V_2) \cup (W_1 \times W_2)$, v and w be as in Example 2.10, and $K := \{0\} \times \{v, w\}$. Then K does not have an open cartesian neighbourhood in U.

[Suppose that $P_1 \times P_2 \subseteq U$ is open and $K \subseteq P_1 \times P_2$. Then $0 \in P_1$ and $v, w \in P_2$. Since $(p^m, v) \to (0, v)$ as $m \to \infty$ and $(p^m, v) \in U$, there is $m \in \mathbb{N}$ such that $(p^m, v) \in P_1 \times P_2$. Thus $p^m \in P_1$ and hence $(p^m, w) \in P_1 \times P_2 \subseteq U$. Now $(\mathbb{Q}_p \times W_2) \cap U = W_1 \times W_2$. As this set does not contain (p^m, w), we have reached a contradiction.]

Alternatively, note that if a locally cartesian set $U \subseteq \mathbb{K}^n$ satisfies the conclusion of [19, Lemma 3.9], then $U^{>\beta<}$ is dense in $U^{<\beta>}$ for each $\beta \in \mathbb{N}_0^n$. We have seen in Example 2.10 that U does not have this property.[8]

4 Proof of Theorem A

If $x \in U$, then $f_x := f(x, \bullet) = f^\vee(x) \in C^\beta(V, E)$. To see this, let $Q \subseteq V$ be a cartesian open subset and $\eta \in \mathbb{N}_0^m$ such that $\eta \leq \beta$. There is an open cartesian subset $P \subseteq U$ such that $x \in P$. It is clear that $(f|_{P\times Q})^{>(0,\eta)<}(x, y) = (f_x|_Q)^{>\eta<}(y)$ for all $y \in Q^{>\eta<}$. Hence $(f|_{P\times Q})^{<(0,\eta)>}(x, \bullet)\colon Q^{<\eta>} \to E$ is a continuous extension for $(f_x|_Q)^{>\eta<}$ and thus f_x is C^β.

We claim: *If $f \in C^{(\alpha,\beta)}(U \times V, E)$, $P \subseteq U$ and $Q \subseteq V$ are cartesian open subsets, $\gamma \in \mathbb{N}_0^n$ with $\gamma \leq \alpha$ and $\eta \in \mathbb{N}_0^m$ with $\eta \leq \beta$, then*

$$(\Delta_{\eta,Q} \circ f^\vee|_P)^{>\gamma<}(x)(y) = f^{>(\gamma,\eta)<}(x, y) \tag{19}$$

[7] In the main cases when U is open [16, 5.12] or cartesian, the assertion is correct.

[8] Given $x \in U^{<\beta>}$, define $K_i := \{x_0^{(i)}, \ldots, x_{\beta_i}^{(i)}\}$ for $i \in \{1, \ldots, n\}$. Then $K := K_1 \times \cdots \times K_n \subseteq U$, by definition of $U^{<\beta>}$. If an open cartesian set $V \subseteq U$ with $K \subseteq V$ exists, then $x \in V^{<\beta>}$, which is in the closure of $V^{>\beta<}$. Hence x is in the closure of $U^{>\beta<}$.

for all $(x,y) \in P^{>\gamma<} \times Q^{>\eta<} = (Q \times P)^{>(\gamma,\eta)<}$, *where* $\Delta_{\eta,Q}$ *denotes the map* $C^\beta(V,E) \to C(Q^{<\eta>}, E)$, $g \mapsto g^{<\eta>}$.

If this is true, holding x fixed we deduce that

$$(\Delta_{\eta,Q} \circ f^\vee)^{>\gamma<}(x)(y) = (f|_{P\times Q})^{<(\gamma,\eta)>}(x,y)$$

for all $(x,y) \in P^{>\gamma<} \times Q^{<\eta>}$ (by continuity). Thus

$$(\Delta_{\eta,Q} \circ f^\vee)^{>\gamma<}(x) = ((f|_{P\times Q})^{<(\gamma,\eta)>})^\vee(x)$$

for all $x \in P^{>\gamma<}$. As the maps $((f|_{P\times Q})^{<(\gamma,\eta)>})^\vee \colon P^{<\gamma>} \to C(Q^{<\eta>}, E)$ are continuous (see Proposition B.15), we deduce that each of the maps $\Delta_{\eta,Q} \circ f^\vee|_P$ is C^α, with

$$(\Delta_{\eta,Q} \circ f^\vee|_P)^{<\gamma>} = ((f|_{P\times Q})^{<(\gamma,\eta>})^\vee\,. \tag{20}$$

Using Lemmas 3.7, 2.28 and 2.27, we see that $f^\vee|_P$ is C^α. Hence $f^\vee$ is C^α, by Definition 2.15 (b). Since $\Delta_{\eta,Q}$ is continuous linear, Lemma 2.26 and (20) show that

$$\Delta_{\eta,Q} \circ ((f^\vee|_P)^{<\gamma>}) = (\Delta_{\eta,Q} \circ f^\vee|_P)^{<\gamma>} = ((f|_{P\times Q})^{<(\gamma,\eta)>})^\vee\,. \tag{21}$$

By Proposition B.15, the map

$$\Psi_{\gamma,\eta,P,Q} \colon C(P^{<\gamma>} \times Q^{<\eta>}, E) \to C(P^{<\gamma>}, C(Q^{<\eta>}, E)), \quad h \mapsto h^\vee$$

is a topological embedding. Let Φ be as in Theorem A and consider the mappings $\Delta_{\gamma,P} \colon C^\alpha(U, C^\beta(V,E)) \to C(P^{<\gamma>}, C^\beta(V,E))$, $g \mapsto g^{<\gamma>}$ and $\Delta_{(\gamma,\eta),P\times Q} \colon C^{(\alpha,\beta)}(U \times V, E) \to C(P^{<\gamma>} \times Q^{<\eta>}, E)$, $h \mapsto h^{<(\gamma,\eta)>}$ (where $(P \times Q)^{<(\gamma,\eta)>} = P^{<\gamma>} \times Q^{<\eta>}$). We can then re-read (21) as

$$C(P^{<\gamma>}, \Delta_{\eta,Q}) \circ \Delta_{\gamma,P} \circ \Phi = \Psi_{\gamma,\eta,P,Q} \circ \Delta_{(\gamma,\eta),P\times Q}. \tag{22}$$

Let $\mathcal{O}$ be the compact-open $C^{(\alpha,\beta)}$-topology on $C^{(\alpha,\beta)}(U \times V, E)$ and $\mathcal{T}$ be the initial topology with respect to Φ. Since $\mathcal{O}$ is initial with respect to the maps $\Delta_{(\gamma,\eta),P\times Q}$ and the topology on $C(P^{<\gamma>} \times Q^{<\eta>}, E)$ is initial with respect to $\Psi_{\gamma,\eta,P,Q}$, the transitivity of initial topologies shows that $\mathcal{O}$ is initial with respect to the maps $\Psi_{\gamma,\eta,P,Q} \circ \Delta_{(\gamma,\eta),P\times Q}$.

The topology on $C^\beta(V,E)$ being initial with respect to the maps $\Delta_{\eta,Q}$, Lemma B.4 shows that the topology on $C(P^{<\gamma>}, C^\beta(V,E))$ is initial with

respect to the maps $C(P^{<\gamma>}, \Delta_{\eta,Q})$. Also, the topology on $C^\alpha(U, C^\beta(V,E))$ is initial with respect to the maps $\Delta_{\gamma,P}$. Hence, by transitivity of initial topologies, $\mathcal{T}$ is the initial topology with respect to the maps

$$C(P^{<\gamma>}, \Delta_{\eta,Q}) \circ \Delta_{\gamma,P} \circ \Phi.$$

By (22), these maps coincide with $\Psi_{\gamma,\eta,P,Q} \circ \Delta_{(\gamma,\eta),P\times Q}$. Comparing with the preceding description of $\mathcal{O}$ as an initial topology, we see that $\mathcal{O} = \mathcal{T}$. Since Φ is also injective, we deduce that Φ is a topological embedding.

If $\mathbb{K}$ is metrizable, we have to show that Φ is surjective. To this end, let $g \in C^\alpha(U, C^\beta(V,E))$. Then $g\colon U \to C^\beta(V,E)$ is continuous (see Remark 2.18), entailing that g is continuous also as a map $U \to C(V,E)$ (see Remark 3.2 (d)). Since $U \times V$ is metrizable and hence a k-space, we deduce with Proposition B.15 that

$$f := g^\wedge\colon U \times V \to E, \quad f(x,y) = g(x)(y)$$

is continuous. Now

$$g_{\gamma,\eta,P,Q} := \Delta_{\eta,Q} \circ \Delta_{\gamma,P}(g) \in C(P^{<\gamma>}, C(Q^{<\eta>}, E)).$$

Because $P^{<\gamma>} \times Q^{<\eta>}$ is metrizable and hence a k-space, Proposition B.15 shows that the associated map

$$g^\wedge_{\gamma,\eta,P,Q}\colon P^{<\gamma>} \times Q^{<\eta>} \to E, \quad (x,y) \mapsto g_{\gamma,\eta,P,Q}(x)(y)$$

is continuous. We claim that

$$(f|_{P\times Q})^{>(\gamma,\eta)<} = g^\wedge_{\gamma,\eta,P,Q}|_{P^{>\gamma<}\times Q^{>\eta<}}. \tag{23}$$

If this is true, then $g^\wedge_{\gamma,\eta,P,Q}$ provides a continuous extension for $(f|_{P\times Q})^{>(\gamma,\eta)<}$ to $P^{<\gamma>} \times Q^{<\eta>} = (P \times Q)^{<(\gamma,\eta)>}$. Hence f is $C^{(\alpha,\beta)}$. Moreover, $\Phi(f) = f^\vee = (g^\wedge)^\vee = g$. Thus, it only remains to prove the claims.

Proof of the first claim. Since $\Delta_{\eta,Q}$ is linear, we have $(\Delta_{\eta,Q} \circ f^\vee|_P)^{>\gamma<} = \Delta_{\eta,Q} \circ (f^\vee|_P)^{>\gamma<}$. Let $y \in Q$. Because the point evaluation $\varepsilon_y\colon C^\beta(V,E) \to E$, $g \mapsto g(y)$ is linear, we have

$$\varepsilon_y \circ (f^\vee|_P)^{>\gamma<} = (\varepsilon_y \circ f^\vee)^{>\gamma<} = (x \mapsto f(x,y))^{>\gamma<}.$$

Hence $(f^\vee|_P)^{>\gamma<}(x)(y) = (\varepsilon_y \circ (f^\vee|_P)^{>\gamma<})(x)$ is given by (4) for $x \in P^{>\gamma<}$, replacing β with γ, V with P and f with $f(\bullet, y)$ there. Holding $x \in P^{>\gamma<}$

fixed in the resulting formula, we use (4) to calculate $\Delta_{\eta,Q}((f^\vee|_P)^{>\gamma<}(x))(y)$ for $y \in Q^{>\eta<}$. We obtain that $\Delta_{\eta,Q}((f^\vee|_P)^{>\gamma<}(x))(y)$ is given by

$$\sum_{j_1=0}^{\gamma_1} \cdots \sum_{j_n=0}^{\gamma_n} \sum_{i_1=0}^{\eta_1} \cdots \sum_{i_m=0}^{\eta_m} \Big(\prod_{k_1 \neq j_1} \frac{1}{x_{j_1}^{(1)} - x_{k_1}^{(1)}} \cdot \ldots \cdot \prod_{k_n \neq j_n} \frac{1}{x_{j_n}^{(n)} - x_{k_n}^{(n)}} \cdot$$
$$\prod_{\ell_1 \neq i_1} \frac{1}{y_{i_1}^{(1)} - y_{\ell_1}^{(1)}} \cdot \ldots \cdot \prod_{\ell_m \neq i_m} \frac{1}{y_{i_m}^{(m)} - y_{\ell_m}^{(m)}} \Big) f(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)}, y_{i_1}^{(1)}, \ldots, y_{i_m}^{(m)}). \quad (24)$$

Using (4) to calculate $(f|_{P\times Q})^{>(\gamma,\eta)<}(x,y)$, we get the same formula.

Proof of the second claim. For $y \in Q$ and $x \in P^{>\gamma<}$, we can calculate $((g|_P)^{>\gamma<}))(x)(y) = (\varepsilon_y \circ ((g|_P)^{>\gamma<}))(x) = (\varepsilon_y \circ g|_P)^{>\gamma<}(x) = (f(\bullet, y)|_P)^{>\gamma<}(x)$ using (4). Holding x fixed, we can apply (4) again and infer that $g^\wedge_{\gamma,\eta,P,Q}(x,y)$ $= (\Delta_{\eta,Q} \circ \Delta_{\gamma,P}(g))(x,y)$ is given for $y \in Q^{>\eta<}$ by (24). Using (4), the same formula is obtained for $(f|_{P\times Q})^{>(\gamma,\eta)<}(x,y)$. This completes the proof of the claim and hence completes the proof of Theorem A. $\square$

5 Proof of Theorem B

The proof is by induction on $n \in \mathbb{N}$.

The case $n = 1$. Thus $\alpha = (r)$ with $r := \alpha_1$. For $E = \mathbb{K}$ and $r < \infty$, the assertion was established in [20, Proposition II.40], and one can proceed analogously if $r < \infty$ and E is an ultrametric Banach space.

Now let E be a locally convex space, $f \in C^r(U,E)$ and $W \subseteq C^r(U,E)$ be an open 0-neighbourhood.

The case $r < \infty$. Let $\widetilde{E}$ be a completion of E with $E \subseteq \widetilde{E}$. Since $C^r(U,E)$ carries the topology induced by $C^r(U,\widetilde{E})$ (see Lemma 3.14), we have $W = \widetilde{W} \cap C^r(U,E)$ for some open 0-neighbourhood $\widetilde{W} \subseteq C^r(U,\widetilde{E})$. Now $\widetilde{E} = \varprojlim E_q$ for certain ultrametric Banach spaces $(E_q, \|.\|_q)$, with continuous linear limit maps $\phi_q \colon \widetilde{E} \to E_q$ with dense image. Then $C^r(U,\widetilde{E}) = \varprojlim C^r(U,E_q)$ with respect to the maps $C^r(U,\phi_q)$ (see Lemma 3.16). After shrinking $\widetilde{W}$ (and W), we may therefore assume that $\widetilde{W} = C^r(U,\phi_q)^{-1}(V)$ for some q and open 0-neighbourhood $V \subseteq C^r(U,E_q)$. By the case of ultrametric Banach spaces, there exists $g \in \mathrm{LocPol}_{\leq r}(U,E_q)$ such that $\pi_q \circ f - g \in V$. Thus, there

are $\ell \in \mathbb{N}$, clopen subsets $U_1, \ldots, U_\ell \subseteq U$ and $c_{i,j} \in E_q$ for $i \in \{0, 1, \ldots, r\}$ and $j \in \{1, \ldots, \ell\}$ such that

$$g(x) = \sum_{i=0}^{r} \sum_{j=1}^{\ell} x^i \mathbf{1}_{U_j}(x) c_{i,j},$$

where $\mathbf{1}_{U_j} \colon U \to \mathbb{K}$ takes the value 1 on U_j, and vanishes elsewhere. Let us write $X^i \colon U \to \mathbb{K}$ for the function $x \mapsto x^i$. By Lemma 3.17, there are open neighbourhoods $Q_{i,j} \subseteq E_q$ of $c_{i,j}$ such that

$$\phi_q \circ f - \sum_{i=0}^{r} \sum_{j=1}^{\ell} X^i \mathbf{1}_{U_j} b_{i,j} \in V,$$

for all $b_{i,j} \in Q_{i,j}$. Since $\phi_q(E)$ is dense in E_q, there exist $a_{i,j} \in E$ such that $\phi_q(a_{i,j}) \in Q_{i,j}$. Hence $h := \sum_{i=0}^{r} \sum_{j=1}^{\ell} X^i \mathbf{1}_{U_j} a_{i,j} \in \mathrm{LocPol}_{\leq r}(U, E)$ and

$$\phi_q \circ (f - h) = \phi_q \circ f - \sum_{i=0}^{r} \sum_{j=1}^{\ell} X^i \mathbf{1}_{U_j} \phi_q(a_{i,j}) \in V.$$

Thus $f - h \in C^r(U, \phi_q)^{-1}(V) = \widetilde{W}$ and thus $f - h \in \widetilde{W} \cap E = W$.

If $r = \infty$, recall from Lemma 3.8 that $C^\infty(U, E) = \varprojlim_{k \in \mathbb{N}_0} C^k(U, E)$. We may therefore assume that $W = V \cap C^\infty(U, E)$ for some $k \in \mathbb{N}_0$ and open 0-neighbourhood $V \subseteq C^k(U, E)$. By the preceding, there is $h \in \mathrm{LocPol}_{\leq k}(U, E)$ such that $f - h \in V$. Then $f - h \in V \cap C^\infty(U, E) = W$.

Induction step. If $n \geq 2$, we write $U = U_1 \times U'$ with $U \subseteq \mathbb{K}$ and $U' \subseteq \mathbb{K}^{n-1}$. We also write $\alpha = (r, \alpha')$ with $r \in \mathbb{N}_0 \cup \{\infty\}$ and $\alpha' \in (\mathbb{N}_0 \cup \{\infty\})^{n-1}$. Let $f \in C^\alpha(U, E)$ and $W \subseteq C^\alpha(U, E)$ be an open 0-neighbourhood. By Theorem A, the map

$$\Phi \colon C^\alpha(U, E) \to C^r(U_1, C^{\alpha'}(U', E)), \quad f \mapsto f^\vee$$

is an isomorphism of topological vector spaces. Hence $W = \Phi^{-1}(V)$ for some open 0-neighbourhood $V \subseteq C^r(U_1, C^{\alpha'}(U', E))$. By induction, there exists $g \in \mathrm{LocPol}_{\leq r}(U_1, C^{\alpha'}(U', E))$ with $\Phi(f) - g \in V$. There are $\ell \in \mathbb{N}$, clopen subsets $R_1, \ldots, R_\ell \subseteq U_1$ and $c_{i,j} \in C^{\alpha'}(U', E)$ for $i \in \{0, 1, \ldots, r\}$ and $j \in \{1, \ldots, \ell\}$ such that

$$g(x) = \sum_{i=0}^{r} \sum_{j=1}^{\ell} x^i \mathbf{1}_{R_j}(x) c_{i,j},$$

using the characteristic function $\mathbf{1}_{R_j}\colon U_1 \to \mathbb{K}$. Write $X^i\colon U_1 \to \mathbb{K}$ for the function $x \mapsto x^i$. By Lemma 3.17, there are open neighhbourhoods $Q_{i,j} \subseteq C^{\alpha'}(U', E)$ of $c_{i,j}$ such that

$$\Phi(f) - \sum_{i=0}^{r} \sum_{j=1}^{\ell} X^i \mathbf{1}_{R_j} b_{i,j} \in V,$$

for all $b_{i,j} \in Q_{i,j}$. By induction, we find $b_{i,j} \in Q_{i,j} \cap \mathrm{LocPol}_{\leq \alpha'}(U', E)$. Then

$$g\colon U \to E, \quad g(x, x') := \sum_{i=0}^{r} \sum_{j=1}^{\ell} x^i \mathbf{1}_{R_j}(x) b_{i,j}(x') \quad \text{for } x \in U_1,\ x' \in U'$$

is an element of $\mathrm{LocPol}_{\leq \alpha}(U, E)$. Since $\Phi(g) = \sum_{i=0}^{r} \sum_{j=1}^{\ell} X^i \mathbf{1}_{R_j} b_{i,j}$, we have $\Phi(f) - \Phi(g) \in V$ and thus $f - g \in W$. This completes the proof. □

6 Proof of Theorem C

For $x \in \mathbb{Z}_p$ and $\nu \in \mathbb{N}_0$, as usual we define

$$\binom{x}{\nu} := \frac{x(x-1)\cdots(x-\nu+1)}{\nu!}$$

(see, e.g., [21, Definition 47.1]). This is a polynomial function in x of degree ν. Since $\binom{x}{\nu} \in \mathbb{N}_0$ if $x \in \mathbb{N}$, the density of $\mathbb{N}$ in $\mathbb{Z}_p$ and continuity of $\binom{\bullet}{\nu}$ entail that

$$\binom{x}{\nu} \in \mathbb{Z}_p \quad \text{for all } x \in \mathbb{Z}_p. \tag{25}$$

For $n \in \mathbb{N}$, $\nu \in \mathbb{N}_0^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n$, we consider the product

$$\binom{x}{\nu} := \binom{x_1}{\nu_1} \cdot \ldots \cdot \binom{x_n}{\nu_n} \in \mathbb{Z}_p.$$

Let $\mathcal{F}_n$ be the set of all finite subsets of $\mathbb{N}_0^n$. We make $\mathcal{F}_n$ a directed set by means of inclusion of sets, $\leq := \subseteq$.

Lemma 6.1 *Let E be a topological vector space over $\mathbb{Q}_p$, $n \in \mathbb{N}$ and $a_\nu \in E$ for $\nu \in \mathbb{N}_0^n$. Assume that, for each $x \in \mathbb{Z}_p^n$, the limit*

$$f(x) := \sum_{\nu \in \mathbb{N}_0^n} \binom{x}{\nu} a_\nu := \lim_{\Phi \in \mathcal{F}_n} \sum_{\nu \in \Phi} \binom{x}{\nu} a_\nu \tag{26}$$

of the net of finite partial sums exists. Then the family $(a_\nu)_{\nu \in \mathbb{N}_0^n}$ is uniquely determined by f; the coefficient a_ν can be recovered via

$$a_\nu = \sum_{\mu \leq \nu} (-1)^{|\nu|-|\mu|} \binom{\nu}{\mu} f(\mu) \tag{27}$$

(*where the summation is over all $\mu \in \mathbb{N}_0^n$ such that $\mu \leq \nu$*). *In particular, a_ν is contained in the additive subgroup of $(E, +)$ generated by $f(\mathbb{N}_0^n)$.*

Proof. We closely follow the treatment of the one-dimensional scalar-valued case in the proof of [21, Theorem 52.1]. For $j \in \{1, \ldots, n\}$, define the operator $\delta_j \colon E^{\mathbb{Z}_p^n} \to E^{\mathbb{Z}_p^n}$ via

$$(\delta_j f)(x) := f(x + e_j) - f(x),$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ with only a non-zero j-th entry. Performing the calculations on [21, pp. 152–153] in each of the variables $x_n, x_{n-1}, \ldots, x_1$ in turn, we obtain

$$\begin{aligned} a_\nu &= (\delta_1^{\nu_1} \cdots \delta_n^{\nu_n} f)(0, \ldots, 0) \\ &= \sum_{\mu_1=0}^{\nu_1} \cdots \sum_{\mu_n=0}^{\nu_n} (-1)^{\nu_1-\mu_1} \cdots (-1)^{\nu_n-\mu_n} f(\mu_1, \ldots, \mu_n), \end{aligned}$$

which can be written more compactly as (27). $\square$

Definition 6.2 If E is a vector space over $\mathbb{Q}_p$ and $f \colon \mathbb{Z}_p^n \to E$ a function, we take (27) as the definition of $a_\nu \in E$ and call a_ν the *ν-th Mahler coefficient of f*. The series (net of finite partial sums)

$$\sum_{\nu \in \mathbb{N}_0^n} \binom{\bullet}{\nu} a_\nu$$

of functions $\mathbb{Z}_p^n \to E$ is called the *Mahler series of f*.

Remark 6.3 Let f be as in Defintion 6.2, $a_\nu \in E$ be its ν-th Mahler coefficient and $\lambda\colon E \to F$ be a continuous linear map to a topological $\mathbb{Q}_p$-vector space F. Applying λ to (27), we see that $\lambda(a_\nu)$ is the ν-th Mahler coefficient of the continuous function $\lambda \circ f\colon \mathbb{Z}_p^n \to F$.

Definition 6.4 Let X be a set, E be a locally convex space over an ultrametric field $\mathbb{K}$ and $w\colon X \to]0,\infty[$ be a function. We define $c_0(X,w,E)$ as the space of all $f\colon X \to E$ such that, for each ultrametric continuous seminorm q on E and $\varepsilon > 0$, there exists a finite subset $\Phi \subseteq X$ such that

$$(\forall x \in X \setminus \Phi) \quad w(x)q(f(x)) < \varepsilon.$$

We endow $c_0(X,w,E)$ with the locally convex topology defined by the ultrametric seminorms given by

$$\|f\|_{w,q} := \sup\{w(x)q(f(x))\colon x \in X\} \in [0,\infty[.$$

If w is the constant function 1, we abbreviate $c_0(X,E) := c_0(X,1,E)$ and $\|.\|_{q,\infty} := \|.\|_{1,q}$.

Lemma 6.5 *Let X, w and E be as in Definition 6.4 and $\lambda\colon E \to F$ be a continuous linear map to a locally convex space F over $\mathbb{K}$. Then $\lambda \circ f \in c_0(X,w,F)$ for each $f \in c_0(X,w,E)$, and the map*

$$c_0(X,w,\lambda)\colon c_0(X,w,E) \to c_0(X,w,F), \quad f \mapsto \lambda \circ f$$

is continuous and linear. If λ is a linear topological embedding, then also $c_0(X,w,\lambda)$.

Proof. If q is an ultrametric continuous seminorm on F, then $q \circ \lambda$ is an ultrametric continuous seminorm on E. If $\varepsilon > 0$, there exists a finite subset $\Phi \subseteq X$ such that $\sup\{w(x)(q\circ\lambda)(f(x))\colon x \in X \setminus \Phi\} < \varepsilon$. Since $(q\circ\lambda)(f(x)) = q((\lambda \circ f)(x))$, we see that $\lambda \circ f \in c_0(X,w,F)$. Because

$$\|\lambda \circ f\|_{w,q} = \|f\|_{w,q\circ\lambda} \leq \|f\|_{w,q\circ\lambda},$$

the linear map $c_0(X,w,\lambda)$ is continuous.

If λ is a topological embedding, then $c_0(X,w,\lambda)$ is injective. Hence $c_0(X,w,\lambda)$ will be an embedding if the initial topology on $c_0(X,w,E)$ with respect to $c_0(X,w,\lambda)$ coincides with the given topology. But this is a special case of the next lemma. $\square$

Lemma 6.6 *Let X, w and E be as in Definition 6.4 and assume that the topology on E is initial with respect to a family $(\lambda_j)_{j\in J}$ of linear mappings $\lambda_j\colon E \to E_j$ to locally convex spaces E_j. Then the topology on $c_0(X,w,E)$ is initial with respect to the family $(c_0(X,w,\lambda_j))_{j\in J}$.*

Proof. By hypothesis, the locally convex vector topology on E can be defined by the seminorms $q\circ\lambda_j$, for $j \in J$ and q ranging through the ultrametric continuous seminorms on E_j. The given locally convex topology $\mathcal{O}$ on $c_0(X,w,E)$ is therefore defined by the seminorms of the form $\|.\|_{w,q\circ\lambda_j}$. On the other hand, the initial topology $\mathcal{T}$ on $c_0(X,w,E)$ with respect to the maps $c_0(X,w,\lambda_j)$ is the locally convex vector topology defined by the ultrametric seminorms $\|.\|_{w,q}\circ c_0(X,w,\lambda_j)$. As these coincide with the seminorms $\|.\|_{w,q\circ\lambda_j}$ from above, $\mathcal{O}=\mathcal{T}$ follows. $\square$

Lemma 6.7 *Let X, w and E be as in Definition 6.4. The locally convex space $c_0(X,w,E)$ has the following properties:*

(a) *If E is metrizable, then $c_0(X,w,E)$ is metrizable.*

(b) *If E is complete (resp., sequentially complete), then also $c_0(X,w,E)$ is complete (resp., sequentially complete).*

(c) *If E is an ultrametric Banach space, then also $c_0(X,w,E)$.*

Proof. (a) If E is metrizable, then its vector topology is defined by a countable set Q of ultrametric seminorms. Then $\{\|.\|_{w,q}\colon q\in Q\}$ is a countable set of ultrametric seminorms that defines the locally convex topology of $c_0(X,w,E)$, and hence $c_0(X,w,E)$ is metrizable.

(b) If E is complete, let $(h_j)_{j\in J}$ be a Cauchy net in $c_0(X,w,E)$. Since the point evaluation $\varepsilon_x\colon c_0(X,w,E)\to E$ is continuous and linear for $x\in X$, we see that $(h_j(x))_{j\in J}$ is a Cauchy net in E and hence convergent to some $h(x)\in E$. Given an ultrametric continuous seminorm q on E and $\varepsilon>0$, there exists $j_0\in J$ such that $\|h_j-h_k\|_{w,q}\le\varepsilon$ for all $j,k\in J$ such that $j,k\ge j_0$. For fixed $x\in X$, we therefore have $w(x)q(h_j(x)-h_k(x))\le\varepsilon$ for all $j,k\in J$ such that $j,k\ge j_0$. Passing to the limit in k, we see that

$$(\forall j\ge j_0)(\forall x\in X)\ \ w(x)q(h_j(x)-h(x))\le\varepsilon. \tag{28}$$

Hence $w(x)q(h(x)) \leq w(x)q(h_{j_0})+\varepsilon$ and thus $h \in c_0(X,w,E)$, with $\|h\|_{w,q} \leq \|h_{j_0}\|_{w,q} + \varepsilon$. Now (28) can be read as $\|h_j - h\|_{w,q} \leq \varepsilon$ for all $j \geq j_0$. Thus $h_j \to h$, showing the completeness of $c_0(X,w,E)$.

(c) If the ultrametric norm $q := \|.\|$ defines the vector topology on E, then $\|.\|_{w,q}$ is an ultrametric norm that defines the vector topology on $c_0(X,w,E)$ and makes it an ultrametric Banach space (by (b)). $\square$

We abbreviate $c_0(X,\lambda) := c_0(X,1,\lambda)$.

Lemma 6.8 *Let X and Y be sets, $v\colon X \to]0,\infty[$ and $w\colon Y \to]0,\infty[$ be functions and E be a locally convex space over an ultrametric field $\mathbb{K}$. Define $v \otimes w\colon X \times Y \to]0,\infty[$ via $(v\otimes w)(x,y) := v(x)w(y)$. Then $f^\vee(x) := f_x := f(x,\bullet) \in c_0(Y,w,E)$ for each $x \in X$ and $f \in c_0(X\times Y, v\otimes w, E)$. Moreover, $f^\vee \in c_0(X,v,c_0(Y,w,E))$, and the map*

$$\Theta\colon c_0(X\times Y, v\otimes w, E) \to c_0(X,v,c_0(Y,w,E)), \quad f \mapsto f^\vee$$

is an isomorphism of topological vector spaces with inverse

$$\Psi\colon c_0(X,v,c_0(Y,w,E)) \to c_0(X\times Y, v\otimes w, E), \quad g \mapsto g^\wedge,$$

where $g^\wedge(x,y) := g(x)(y)$.

Proof. Let q be an ultrametric continuous seminorm on E and $\varepsilon > 0$. Fix $z \in X$. There exists a finite subset $\Phi \subseteq X \times Z$ such that $v(x)w(y)q(f(x,y)) < \min\{\varepsilon, v(z)\varepsilon\}$ for all $(x,y) \in (X\times Y)\setminus\Phi$. After increasing Φ, we may assume that $\Phi = \Phi_1 \times \Phi_2$ with finite subsets $\Phi_1 \subseteq X$, $\Phi_2 \subseteq Y$. For $y \in Y \setminus \Phi_2$, we then have $w(y)q(f(z,y)) < \varepsilon$, and we deduce that $f_z \in c_0(Y,w,E)$. Moreover, for all $x \in E \setminus \Phi_1$ and all $y \in Y$ we have $v(x)w(y)q(f(x,y)) < \varepsilon$. Passing to the supremum over y, we obtain $v(x)\|f_x\|_{w,q} \leq \varepsilon$. Hence $f^\vee \in c_0(X,v,c_0(Y,w,E))$ indeed. Because

$$\begin{aligned}\|f\|_{v\otimes w,q} &= \sup\{v(x)w(y)q(f(x,y))\colon x\in X, y \in Y\}\\ &= \sup_{x\in X} v(x) \sup_{y\in Y} w(y)q(f(x,y)) = \sup_{x\in X} v(x)\|f_x\|_{w,q} = \|f^\vee\|_{v,\|.\|_{w,q}},\end{aligned}$$

the injective linear map Θ is a topological embedding. To see that Θ is surjective, let $g \in c_0(X,v,c_0(Y,w,E))$. Define $g^\wedge\colon X\times Y \to E$, $g^\wedge(x,y) := g(x)(y)$. If we can show that $g^\wedge \in c_0(X\times Y, v\otimes w, E)$, then $\Theta(g^\wedge) = (g^\wedge)^\vee = g$ proves the surjectivity. To this end, let q and ε be as before.

Then there is a finite subset $\Phi_1 \subseteq X$ such that $v(x)\|g(x)\|_{w,q} < \varepsilon$ for all $x \in X \setminus \Phi_1$. For each $x \in \Phi_1$, there exists a finite subset $\Xi_x \subseteq Y$ such that $w(x)v(y)q(g(x)(y)) < \varepsilon$ for all $y \in Y \setminus \Xi_x$. Set $\Phi_2 := \bigcup_{x\in\Phi_1} \Xi_x$ and $\Phi := \Phi_1 \times \Phi_2$. If $(x,y) \in (X \times Y) \setminus \Phi$, then $x \in X \setminus \Phi_1$ (in which case $v(x)w(y)q(g(x)(y)) \leq v(x)\|g(x)\|_{w,q} < \varepsilon$) or $x \in \Phi_1$ and $y \in Y \setminus \Phi_2$, in which case $y \in Y \setminus \Xi_x$ and thus $w(x)v(y)q(g(x)(y)) < \varepsilon$ as well. Thus $g^\wedge \in c_0(X \times Y, v \otimes w, E)$ indeed. $\square$

If X is a compact space, E a locally convex space over an ultrametric field and q an ultrametric continuous seminorm on E, we obtain an ultrametric continuous seminorm on $C(X,E)$ via

$$\|f\|_{q,\infty} := \sup\{q(f(x)) \colon x \in X\} \in [0,\infty[\quad \text{for } f \in C(X,E).$$

Lemma 6.9 *Let E be a locally convex space over $\mathbb{Q}_p$ and $f \colon \mathbb{Z}_p^n \to E$ be a continuous function, with Mahler coefficients $a_\nu(f)$ for $\nu \in \mathbb{N}_0^n$. Then the Mahler series of f converges uniformly to f (i.e., it converges to f in $C(\mathbb{Z}_p^n, E)$). Further, $(a_\nu(f))_{\nu\in\mathbb{N}_0^n} \in c_0(\mathbb{N}_0^n, E)$, and the map*

$$A_{n,E} := (a_\nu)_{\nu\in\mathbb{N}_0^n} \colon C(\mathbb{Z}_p^n, E) \to c_0(\mathbb{N}_0^n, E), \quad f \mapsto (a_\nu(f))_{\nu\in\mathbb{N}_0^n}$$

is a linear topological embedding with

$$\|A_{n,E}(f)\|_{q,\infty} = \|f\|_{q,\infty} \tag{29}$$

for each continuous ultrametric seminorm q on E. If E is sequentially complete, then $A_{n,E}$ is an isomorphism of topological vector spaces.

Proof. Let $\widetilde{E}$ be a completion of E with $E \subseteq \widetilde{E}$. The proof is by induction on $n \in \mathbb{N}$. If $n = 1$, then one finds as in [21, p. 156, Exercise 52E(v)][9] that $q(a_\nu(f)) \to 0$ as $\nu \to \infty$, for each $f \in C(\mathbb{Z}_p, E)$ and continuous ultrametric seminorm q on E. Hence $A_{1,E}(f) \in c_0(\mathbb{Z}_p, E)$. Since

$$q\left(\binom{x}{\nu} a_\nu(f)\right) \leq q(a_\nu(f)), \tag{30}$$

we deduce that the Mahler series of f converges uniformly in $C(\mathbb{Z}_p, \widetilde{E})$, to a continuous function $g \colon \mathbb{Z}_p \to \widetilde{E}$. As in the cited exercise, one finds that

[9] Due to a misprint, the exercise is labelled Exercice 52.D, like the previous one.

$f(x) = g(x)$ for $x \in \mathbb{N}_0$ and hence for all $x \in \mathbb{Z}_p$, by continuity. The Mahler series thus converges uniformly to f. We now deduce from (30) and (26) that

$$\|f\|_{q,\infty} \le \sup\{q(a_\nu(f)) \colon \nu \in \mathbb{N}_0\} = \|A_{1,E}(f)\|_{q,\infty}.$$

Conversely, (27) entails that $q(a_\nu(f)) \le \|f\|_{q,\infty}$ for each $\nu \in \mathbb{N}_0$. Hence also $\|A_{1,E}(f)\|_{q,\infty} \le \|f\|_{q,\infty}$, and (29) follows. It is clear that $A_{1,E}$ is linear, and Lemma 6.1 implies that $A_{1,E}$ is injective (noting that f is given by its Mahler series, as just shown). Summing up, $A_{1,E}$ is a linear topological embedding.

If E is sequentially complete, let $b = (b_\nu)_{\nu \in \mathbb{N}_0} \in c_0(\mathbb{N}_0, E)$. Then the limit $f(x) := \sum_{\nu \in \mathbb{N}_0} \binom{x}{\nu} b_\nu$ exists in E, uniformly in x, and hence defines a continuous function $f \colon \mathbb{Z}_p \to E$. By Lemma 6.1, $A_{1,E}(f) = (b_\nu)_{\nu \in \mathbb{N}_0}$. Thus $A_{1,E}$ is also surjective and hence an isomorphism of topological vector spaces.

Induction step. If $n \ge 2$, abbreviate $F := C(\mathbb{Z}_p^{n-1}, E)$. This space is sequentially complete if E is sequentially complete, by Lemma B.21 (e). Now $f^\vee \colon \mathbb{Z}_p \to C(\mathbb{Z}_p^{n-1}, E) = F$, $x \mapsto f_x := f(x, \bullet)$ is continuous, for each $f \in C(\mathbb{Z}_p^n, E)$, and the map

$$\Phi \colon C(\mathbb{Z}_p^n, E) \to C(\mathbb{Z}_p, F), \quad f \mapsto f^\vee$$

is an isomorphism of topological vector spaces (see Proposition B.15). By the case $n = 1$, the map

$$A_{1,F} \colon C(\mathbb{Z}_p, F) \to c_0(\mathbb{N}_0, F)$$

is a linear topological embedding (and an isomorphism if E and hence F is sequentially complete). Since $A_{n-1,E}$ is a linear topological embedding by induction (and an isomorphism if E is sequentially complete), also the map

$$c_0(\mathbb{N}_0, A_{n-1,E}) \colon c_0(\mathbb{N}_0, F) \to c_0(\mathbb{N}_0, c_0(\mathbb{N}_0^{n-1}, E))$$

is a topological embedding (resp., an isomorphism), by Lemma 6.5. Finally,

$$\Psi \colon c_0(\mathbb{N}_0, c_0(\mathbb{N}_0^{n-1}, E)) \to c_0(\mathbb{N}_0^n, E), \quad g \mapsto g^\wedge$$

(with $g^\wedge(\nu_1, \ldots, \nu_n) := g(\nu_1)(\nu_2, \ldots, \nu_n)$) is an isomorphism of topological vector spaces, by Lemma 6.8. Hence $\Psi \circ c_0(\mathbb{N}_0, A_{n-1,E}) \circ A_{1,F} \circ \Phi$ is a linear

topological embedding (and an isomorphism if E is sequentially complete). We claim that

$$\Psi \circ c_0(\mathbb{N}_0, A_{n-1,E}) \circ A_{1,F} \circ \Phi = A_{n,E}. \tag{31}$$

If this is true, then all assertions are verified, except for (29). However, (29) readily follows from (27) and (30), as in the case $n = 1$.

To verify the claim, let $\nu = (\nu_1, \nu')$ with $\nu_1 \in \mathbb{N}_0$ and $\nu' \in \mathbb{N}_0^{n-1}$. For $\mu' \in \mathbb{N}_0^{n-1}$ such that $\mu' \leq \mu$, the point evaluation $\varepsilon_{\mu'} \colon C(\mathbb{N}_0^{n-1}, E) \to E$, $g \mapsto g(\mu')$ is continuous linear. Using Remark 6.3, we deduce that

$$\begin{aligned} A_{1,F}(f^\vee)(\nu_1)(\mu') &= \varepsilon_{\mu'}(A_{1,F}(f^\vee)(\nu_1)) \;=\; A_{1,E}((\varepsilon_{\mu'} \circ f^\vee)(\nu_1) \\ &= A_{1,E}(f(\bullet, \mu'))(\nu_1) \\ &= \sum_{\mu_1=0}^{\nu_1} (-1)^{\nu_1 - \mu_1} \binom{\nu_1}{\mu_1} f(\mu_1, \mu'). \end{aligned} \tag{32}$$

As a consequence,

$$\begin{aligned} &(\Psi \circ c_0(\mathbb{N}_0, A_{n-1,E}) \circ A_{1,F} \circ \Phi)(f)(\nu) \;=\; \Psi(A_{n-1,E} \circ A_{1,F}(f^\vee))(\nu) \\ &\quad= (A_{n-1,E} \circ A_{1,F}(f^\vee))(\nu_1)(\nu') \;=\; A_{n-1,E}(A_{1,F}(f^\vee)(\nu_1))(\nu') \\ &\quad= \sum_{\mu' \leq \nu'} (-1)^{|\nu'| - |\mu'|} \binom{\nu'}{\mu'} A_{1,F}(f^\vee)(\nu_1)(\mu') \\ &\quad= \sum_{\mu \leq \nu} (-1)^{|\nu| - |\mu|} \binom{\nu}{\mu} f(\mu) \;=\; A_{n,E}(f)(\nu), \end{aligned}$$

using (32) to obtain the penultimate equality. $\square$

6.10 For $\alpha \in \mathbb{N}_0^n$, define $w_\alpha \colon \mathbb{N}_0^n \to \,]0, \infty[$ via $w_\alpha(\nu) := \nu^\alpha := \nu_1^{\alpha_1} \cdots \nu_n^{\alpha_n}$.

Proposition 6.11 *In the situation of Theorem C, the map*

$$A_E^\alpha \colon C^\alpha(\mathbb{Z}_p^n, E) \to c_0(\mathbb{N}_0^n, w_\alpha, E)$$

taking a function to its Mahler coefficients is an isomorphism of topological vector spaces.

Proof of Theorem C and Proposition 6.11. The final assertion of Theorem C is a special case of Corollary 9.3 (see also Remark 9.4), whence we omit its proof here. The other assertions are proved by induction on $n \in \mathbb{N}$.

The case $n = 1$. Write $\alpha = (r)$ with $r \in \mathbb{N}_0$. If $f \in C(\mathbb{Z}_p, E)$, let $(a_\nu(f))_{\nu \in \mathbb{N}_0}$ be the sequence of Mahler coefficients of f. Using ultrametric continuous seminorms instead of $|.|$, we can repeat the proof of [21, Theorem 54.1] (devoted the scalar-valued case) and find that f is C^r if and only if $q(a_\nu(f))\nu^r \to 0$ as $\nu \to \infty$, for each ultrametric continuous seminorm q on E. In particular, we can define a map

$$A_E^r \colon C^r(\mathbb{Z}_p, E) \to c_0(\mathbb{N}, w_r, E), \quad f \mapsto (a_\nu(f))_{\nu \in \mathbb{N}_0}.$$

It is clear that A_E^r is linear, and it is injective (by Lemma 6.1). To see that A_E^r is also surjective (and hence an isomorphism of vector spaces), let $b = (b_\nu)_{\nu \in \mathbb{N}_0} \in c_0(\mathbb{N}, w_r, E)$. Since $w_r \geq 1$, we have $b \in c_0(\mathbb{N}, E)$. As E is assumed sequentially complete, this entails the existence of a continuous function $f \colon \mathbb{Z}_p \to E$ with Mahler coefficients b_ν (Lemma 6.9). But then $f \in C^r(\mathbb{Z}_p, E)$ by the characterization of C^r-functions just obtained, and now $b = A_E^r(f)$ by construction of f.

To see that A_E^1 is an isomorphism of topological vector spaces, it only remains to show that A_E^1 is a topological embedding. We may assume that E is complete.[10] Thus $E = \varprojlim E_q$ for some ultrametric Banach spaces E_q, with limit maps $\pi_q \colon E \to E_q$. Then the topology on $C^r(\mathbb{Z}_p, E)$ is initial with respect to the maps $C^r(\mathbb{Z}_p, \pi_q)$ (Lemma 3.13) and the topology on $c_0(\mathbb{N}_0, w_r, E)$ is initial with respect to the maps $c_0(\mathbb{N}_0, w_r, \pi_q)$ (Lemma 6.6). Hence, using the transitivity of initial topologies, the initial topology $\mathcal{T}$ on $C^r(\mathbb{Z}_p, E)$ with respect to the map A_E^1 is also initial with respect to the maps $c_0(\mathbb{N}_0, w_r, \pi_q) \circ A_E^1 = A_{E_q}^1 \circ C^r(\mathbb{Z}_p, \pi_q)$ (where the equality of both sides comes from Remark 6.3). If we can show that each $A_{E_q}^1$ is an embedding, then the topology on $C^r(\mathbb{Z}_p, E_q)$ is initial with respect to $A_{E_q}^1$ and hence the compact-open C^r-topology $\mathcal{O}$ on $C^r(\mathbb{Z}_p, E)$ is initial with respect to the maps $A_{E_q}^1 \circ C^r(\mathbb{Z}_p, \pi_q)$. It therefore coincides with $\mathcal{T}$, whence the bijective map A_E^1 is a homeomorphism (and hence an isomorphism of topological vector spaces).

[10] If $\widetilde{E}$ is a completion of E containing E, then $C^r(\mathbb{Z}_p, E)$ embeds into $C^r(\mathbb{Z}_p, \widetilde{E})$ (see Lemma 3.12) and $c_0(\mathbb{N}_0, w_r, E)$ embeds into $c_0(\mathbb{N}_0, w_r, E)$ (Lemma 6.5), whence A_E^1 will be an embedding if $E_{\widetilde{E}}^1$ is so.

By the preceding, it suffices to show that A_E^1 is an isomorphism of topological vector spaces if E is an ultrametric Banach space. In this case, $C^r(\mathbb{Z}_p, E)$ is an ultrametric Banach space (Lemma 3.11 (f)). Also $c_0(\mathbb{N}_0, w_r, E)$ is an ultrametric Banach space (Lemma 6.7). Since A_E^1 is an isomorphism of vector spaces, the Closed Graph Theorem (see, e.g., [22, Proposition 8.5]) implies that A_E^1 will be an isomorphism of topological vector spaces if its graph

$$\Gamma := \{(f, A_E^1(f)) \colon f \in C^r(\mathbb{Z}_p, E)\}$$

is closed in $C^r(\mathbb{Z}_p, E) \times c_0(\mathbb{N}_0, w_r, E)$. To verify the latter, let $(f_i)_{i \in \mathbb{N}}$ be a sequence in $C^r(\mathbb{Z}_p, E)$ which converges to some $f \in C^r(\mathbb{Z}_p, E)$, such that $A_E^1(f_i)$ converges in $c_0(\mathbb{N}_0, w_r, E)$ to some $b = (b_\nu)_{\nu \in \mathbb{N}_0} \in c_0(\mathbb{N}_0, w_r, E)$. The topology on $c_0(\mathbb{N}_0, w_r, E)$ being finer than the topology of pointwise convergence, the point evaluation

$$\varepsilon_\nu \colon c_0(\mathbb{N}_0, w_r, E) \to E, \quad (c_\mu)_{\mu \in \mathbb{N}_0} \mapsto c_\nu$$

is continuous, for each $\nu \in \mathbb{N}_0$. Hence $A_E^1(f_i) \to b$ implies $a_\nu(f_i) \to b_\nu$. But $a_\nu(f_i) \to a_\nu(f)$, as is clear from (27). Hence f has the Mahler coefficients b_ν and thus $(f, b) = (f, A_E^1(f)) \in \Gamma$, proving that Γ is closed.

Induction step. If $n \geq 2$, write $\alpha = (\alpha_1, \alpha')$ with $\alpha_1 \in \mathbb{N}_0$ and $\alpha' \in \mathbb{N}_0^{n-1}$. Since E is sequentially complete and $\mathbb{Q}_p$ is metrizable, $H := C^{\alpha'}(\mathbb{Z}_p^{n-1}, E)$ is sequentially complete (Lemma 3.11 (b)). The map

$$\Phi_\alpha \colon C^\alpha(\mathbb{Z}_p^n, E) \to C^{\alpha_1}(\mathbb{Z}_p, C^{\alpha'}(\mathbb{Z}_p^{n-1}, E)), \quad f \mapsto f^\vee$$

is an isomorphism of topological vector spaces, by Theorem A. Since H is sequentially complete, $A_H^{\alpha_1} \colon C^{\alpha_1}(\mathbb{Z}_p, C^{\alpha'}(\mathbb{Z}_p^{n-1}, E)) \to c_0(\mathbb{N}_0, w_{\alpha_1}, C^{\alpha'}(\mathbb{Z}_p^{n-1}, E))$ is an isomorphism of topological vector spaces (by the case $n = 1$). Because E is sequentially complete, the map $A_E^{\alpha'} \colon C^{\alpha'}(\mathbb{Z}_p^{n-1}, E) \to c_0(\mathbb{N}_0^{n-1}, w_{\alpha'}, E)$ is an isomorphism of topological vector spaces, by induction. Finally, the map $\Psi_\alpha \colon c_0(\mathbb{N}_0, w_{\alpha_1}, c_0(\mathbb{N}_0^{n-1}, w_{\alpha'}, E)) \to c_0(\mathbb{N}_0^n, w_\alpha, E)$, $h \mapsto h^\wedge$ is an isomorphism of topological vector spaces, by Lemma 6.8 (using that $w_{\alpha_1} \otimes w_{\alpha'} = w_\alpha$). Hence also the map

$$\Theta_\alpha := \Psi_\alpha \circ c_0(\mathbb{N}_0, w_{\alpha_1}, A_E^{\alpha'}) \circ A_H^{\alpha_1} \circ \Phi_\alpha \colon C^\alpha(\mathbb{Z}_p^n, E) \to c_0(\mathbb{N}_0^n, w_\alpha, E)$$

is an isomorphism of topological vector spaces. Note that Θ_α is a restriction of the composition $\Theta_0 = A_{n,E}$ considered in (31). Hence $\Theta_\alpha(f)$ is the

family of Mahler coefficients of f, if $f \in C^\alpha(\mathbb{Z}_p^n, E)$. Thus $(a_\nu(f))_{\nu\in\mathbb{N}_0^n} = \Theta_\alpha(f) \in c_0(\mathbb{N}_0^n, w_\alpha, E)$ in this case, i.e., (1) holds. Conversely, assume that $f \in C(\mathbb{Z}_p^n, E)$ and (1) is satisfied, i.e., $b := (a_\nu(f))_{\nu\in\mathbb{N}_0^n} \in c_0(\mathbb{N}_0^n, w_\alpha, E)$. Then $g := \Theta_\alpha^{-1}(b) \in C^\alpha(\mathbb{Z}_p^n, E)$. Since f and g have the same Mahler coefficients, $f = g \in C^\alpha(\mathbb{Z}_p^n, E)$ follows. □

7 C^α-maps on subsets of $\mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_\ell}$

In this section, $\mathbb{K}$ is a topological field and E a topological $\mathbb{K}$-vector space.

Definition 7.1 Let $n_1, \ldots, n_\ell \in \mathbb{N}$, $n := n_1 + \cdots + n_\ell$, $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$ and $U \subseteq \mathbb{K}^n$ be a locally cartesian subset. We let $C^\alpha(U, E)$ be the space of all C^α-maps $f\colon U \to E$ (as defined before Theorem D). Thus

$$C^\alpha(U, E) = \bigcap_{\beta\in N_\alpha} C^\beta(U, E), \tag{33}$$

where N_α is the set of all $\beta = (\beta_1, \ldots, \beta_\ell) \in \mathbb{N}_0^n = \mathbb{N}_0^{n_1} \times \cdots \times \mathbb{N}_0^{n_\ell}$ such that $|\beta_j| \leq \alpha_j$ for all $j \in \{1, \ldots, \ell\}$. We endow $C^\alpha(U, E)$ with the initial topology $\mathcal{O}$ with respect to the inclusion maps

$$\iota_\beta\colon C^\alpha(U, E) \to C^\beta(U, E),$$

using the compact-open C^β-topology on the right-hand side. We call $\mathcal{O}$ the compact-open C^α-topology on $C^\alpha(U, E)$.

If $n_j = 1$ for each j, then the new definition of the compact-open C^α-topology on $C^\alpha(U, E)$ coincides with the earlier one (cf. Lemma 3.8).

Taking $\ell = 1$, we have $\alpha = (r)$ with some $r \in \mathbb{N}_0 \cup \{\infty\}$. Then the $C^{(r)}$-maps are precisely the C^r-maps, defined as follows:

Definition 7.2 Let $\mathbb{K}$ be a topological field, $U \subseteq \mathbb{K}^n$ be a locally cartesian subset, E a topological $\mathbb{K}$-vector space and $r \in \mathbb{N}_0 \cup \{\infty\}$. A map $f\colon U \to E$ is called C^r if it is C^α for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq r$.

Remark 7.3 In the cases when U is open or U is cartesian, this corresponds to the Definition of C^r_{SDS}-maps given by the author in [11] (based on ideas from [21] and [7]). In this case, the compact-open C^r-topology (without that name) was already introduced in the preprint version of [11] (see Definition B.1 in `arXiv:math/0609041v1`), and (for U open) shown to coincide with the topology on $C^r(U, E)$ introduced earlier in [13][11] (see

[11]The latter is even defined if U is an open set in an arbitrary topological vector space.

Theorem B.2 in the preprint version of [11]). The step to locally cartesian sets was performed in [19].

Lemma 7.4 *The mapping*

$$\iota := (\iota_\beta)_{\beta\in N_\beta} \colon C^\alpha(U,E) \to \prod_{\beta\in N_\beta} C^\beta(U,E) \tag{34}$$

is a linear topological embedding with closed image.

Proof. Because the topology on $P := \prod_{\beta\in N_\beta} C^\beta(U,E)$ is initial with the respect to the projections $\mathrm{pr}_j \colon P \to C^\beta(U,E)$, the initial topology $\mathcal{T}$ on $C^\alpha(U,E)$ with respect to ι coincides with the initial topology with respect to the maps $\mathrm{pr}_j \circ \iota = \iota_\beta$ (by transitivity of initial topologies), and hence with $\mathcal{O}$. Thus, being also injective, ι is a topological embedding. The linearity is clear. For each $\beta \in N_\alpha$, the inclusion map $\iota_{0,\beta} \colon C^\beta(U,E) \to C(U,E)$ is continuous. Since

$$\begin{aligned} \mathrm{im}(\iota) &= \{(f_\eta)_{\eta\in N_\alpha} \colon (\forall \beta,\gamma \in N_\alpha)\ f_\beta = f_\gamma\} \\ &= \bigcap_{\beta,\gamma\in N_\alpha} \{f \in P \colon (\iota_{0,\beta}\circ \mathrm{pr}_\beta)(f) = (\iota_{0,\gamma}\circ \mathrm{pr}_\gamma)(f)\}, \end{aligned}$$

$\mathrm{im}(\iota)$ is closed. □

Remark 7.5 Let $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$ and $U \subseteq \mathbb{K}^n$ be a locally cartesian subset with $n = n_1 + \cdots + n_\ell$. It is essential for the proofs of Propositions 9.11 and 9.12 that various results from Section 3 remain valid for the more general C^α-maps on U in the sense of Definition 7.1:

(a) Lemma 3.5 remains valid.

[For $\beta \in N_\alpha$, let $\iota_\beta \colon C^\alpha(U,E) \to C^\beta(U,E)$ and $\iota_\beta^S \colon C^\alpha(S,E) \to C^\beta(S,E)$ be the inclusion maps and $\rho_\beta \colon C^\beta(U,E) \to C^\beta(S,E)$ be the restriction map, which is continuous linear by Lemma 3.5. Then $\iota_\beta^S \circ \rho = \rho_\beta \circ \iota_\beta$ is continuous and hence also ρ.]

(b) Lemma 3.6 remains valid.

[Let ρ_a be as in the lemma, $\rho_{\beta,a} \colon C^\beta(U,E) \to C^\beta(U_a,E)$ be the restriction map and $\iota_{\beta,a} \colon C^\alpha(U_a,E) \to C^\beta(U,E)$ the inclusion map for

$\beta \in N_\alpha$, $a \in A$. As a consequence of Lemma 3.6, the compact-open C^α-topology on $C^\alpha(U,E)$ is initial with respect to the maps $\rho_{\beta,a} \circ \iota_\beta = \iota_{\beta,a} \circ \rho_a$ and hence also with respect to the maps ρ_a.]

(c) Lemma 3.8 carries over in the sense that $C^\alpha(U,E) = \varprojlim_{\beta \in B_\alpha} C^\beta(U,E)$, where B_α is the set of all $\beta \in \mathbb{N}_0^\ell$ with $\beta \leq \alpha$.

[Consider the inclusion maps $\iota_\gamma \colon C^\alpha(U,E) \to C^\gamma(U,E)$ for $\gamma \in N_\alpha$, as well as the inclusion maps $\iota_{\beta,\gamma} \colon C^\gamma(U,E) \to C^\beta(U,E)$ for $\gamma \in N_\alpha$ and $\beta \in B_\gamma$. We have $C^\alpha(U,E) = \bigcap_{\beta \in B_\alpha} C^\beta(U,E)$, and the initial topology on $C^\alpha(U,E)$ with respect to the inclusion maps $\phi_\beta \colon C^\alpha(U,E) \to C^\beta(U,E)$ for $\beta \in B_\alpha$ coincides with the initial topology with respect to the maps $\iota_{\gamma,\beta} \circ \phi_\beta = \iota_\gamma$, for $\beta \in B_\alpha$ and $\gamma \in N_\beta \subseteq N_\alpha$. Since $\bigcup_{\beta \in B_\alpha} N_\beta = N_\alpha$, this is the compact-open C^α-topology.]

(d) Lemma 3.12 remains valid.

[For $\beta \in N_\alpha$, consider the inclusion maps $\iota^E_\beta \colon C^\alpha(U,E) \to C^\beta(U,E)$ and $\iota^F_\beta \colon C^\alpha(U,F) \to C^\beta(U,F)$. Since $\iota^F_\beta \circ C^\alpha(U,\lambda) = C^\beta(U,\lambda) \circ \iota^E_\beta$ is continuous by Lemma 3.12, $C^\alpha(U,\lambda)$ is continuous. If λ is an embedding, then the topology on $C^\beta(U,E)$ is initial with respect to $C^\beta(U,\lambda)$, whence the compact-open C^α-topology on $C^\alpha(U,E)$ is initial with respect to the maps $C^\beta(U,\lambda) \circ \iota^E_\beta = \iota^F_\beta \circ C^\alpha(U,\lambda)$, which is also initial with respect to $C^\alpha(U,\lambda)$. The latter map being also injective, it is a topological embedding.]

(e) Lemma 3.15 remains valid.

[For $\beta \in N_\alpha$, consider the inclusion maps $\iota^E_\beta \colon C^\alpha(U,E) \to C^\beta(U,E)$ and $\iota^{E_j}_\beta \colon C^\alpha(U,E_j) \to C^\beta(U,E_j)$. The initial topology on $C^\alpha(U,E)$ with respect to the maps $C^\alpha(U,\mathrm{pr}_j)$ coincides with the initial topology with respect to the maps $\iota^{E_j}_\beta \circ C^\alpha(U,\mathrm{pr}_j) = C^\beta(U,\mathrm{pr}_j) \circ \iota^E_\beta$. In view of Lemma 3.15, it is therefore also initial with respect to the maps ι^E_β, and hence coincides with the compact-open C^α-topology.]

(f) Lemma 3.16 remains valid.

[Using (d) and (e) from this remark instead of Lemmas 3.14 and 3.15, we can repeat the proof of Lemma 3.16.]

(g) Lemma 3.17 remains valid.

[Let θ be as in the lemma and $\theta_\beta\colon E \to C^\beta(U,E)$ be the corresponding map for $\beta \in N_\alpha$. Then $\iota_\beta \circ \theta = \theta_\beta$ is continuous, by Lemma 3.17. Hence θ is continuous.]

Lemma 7.6 *Let U be a locally cartesian subset of $\mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_\ell} = \mathbb{K}^n$ with $n_1, \ldots, n_\ell \in \mathbb{N}$ and $n = n_1 + \cdots + n_\ell$, and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$. Let $\mathcal{V}$ be a cover of U by cartesian open subsets $V \subseteq U$, and $\mathcal{U}$ be a basis of open 0-neighbourhoods of E. Then a basis of open 0-neighbourhoods in $C^\alpha(U,E)$ is given by finite intersections of sets of the form*

$$\{f \in C^\alpha(U,E)\colon (f|_V)^{<\beta>} \in \lfloor K, Q \rfloor\} \tag{35}$$

with $V \in \mathcal{V}$, $Q \in \mathcal{U}$, $\beta \in N_\alpha$ and compact subsets $K \subseteq V^{<\beta>}$ of the form $K = K_1^{1+\beta_1} \times \cdots \times K_n^{1+\beta_n}$ with compact subsets $K_1, \ldots, K_n \subseteq \mathbb{K}$.

Proof. Step 1. Let $V = V_1 \times \cdots \times V_n$ be an open cartesian subset of U and $\beta \in N_\alpha$. For $i \in \{1, \ldots, n\}$, let $\pi_i\colon \mathbb{K}^{1+\beta_1} \times \cdots \times \mathbb{K}^{1+\beta_n} \to \mathbb{K}^{1+\beta_i}$ be the projection onto the i-th component. For $j \in \{1, \ldots, 1+\beta_i\}$, let $\pi_{j,i}\colon \mathbb{K}^{1+\beta_i} \to \mathbb{K}$ be the projection onto the j-th component. If K is a compact subset of $V^{<\beta>} = V_1^{1+\beta_1} \times \cdots \times V_n^{1+\beta_n}$, then also the larger set $K_1^{1+\beta_1} \times \cdots \times K_n^{1+\beta_n} \subseteq V^{<\beta>}$ is compact, where $K_i := \bigcup_{j=1}^{1+\beta_i} \pi_{j,i}(\pi_i(K))$.

Step 2. In view of Step 1 and Lemma B.21 (b), the sets $\lfloor K, Q \rfloor$ with $Q \in \mathcal{U}$ and $K \subseteq V^{<\beta>}$ a compact set of the form $K_1^{1+\beta_1} \times \cdots \times K_n^{1+\beta_n}$ form a basis $\mathcal{U}_{\beta,V}$ of open 0-neighbourhoods in $C(V^{<\beta>}, E)$.

Step 3. As a consequence of Definition 7.1 and Lemma 3.3, the topology on $C^\alpha(U,E)$ is initial with respect to the maps $\Delta_{\beta,V} \circ \iota_\beta$ with $\beta \in N_\alpha$ and $V \in \mathcal{V}$, whence the injective map

$$\Delta\colon C^\alpha(U,E) \to \prod_{\beta \in N_\alpha, V \in \mathcal{V}} C(V^{<\beta>}, E), \quad f \mapsto (\Delta_{\beta,V}(f))_{\beta \in N_\alpha, V \in \mathcal{V}}$$

is a topological embedding. A basis of open 0-neighbourhoods of the direct product is given by sets of the form $W = \prod_{\beta,V} W_{\beta,V}$, where $W_{\beta,V} = C(V^{<\beta>}, E)$ for all but finitely many (β, V), and the remaining ones are in the basis $\mathcal{U}_{\beta,V}$ of open 0-neighbourhoods in $C(V^{<\beta>}, E)$. Because Δ is a topological embedding, it follows that the pre-images $\Delta^{-1}(W)$ form a basis of 0-neighbourhoods in $C^\alpha(U,E)$. These are the finite intersections described in the lemma. $\square$

8 Proof of Theorem D

For $\gamma \in N_\alpha$ and $\eta \in N_\beta$, the map

$$\Phi_{\gamma,\eta} \colon C^{(\gamma,\eta)}(U \times V, E) \to C^\gamma(U, C^\eta(V, E)), \quad f \mapsto f^\vee$$

is a linear topological embedding, by Theorem A. Because the topology on $C^\beta(V, E)$ is initial with respect to the inclusion maps

$$\iota_\eta \colon C^\beta(V, E) \to C^\eta(V, E),$$

the topology on $C^\gamma(U, C^\eta(V, E))$ is initial with respect to the (inclusion) maps

$$C^\gamma(U, \iota_\eta) \colon C^\gamma(U, C^\beta(V, E)) \to C^\gamma(U, C^\eta(V, E))$$

(see Lemma 3.13). By definition, the topology on $C^{(\alpha,\beta)}(U \times V, E)$ is initial with respect to the inclusion maps $\lambda_{\gamma,\eta} \colon C^{(\alpha,\beta)}(U \times V, E) \to C^{(\gamma,\eta)}(U \times V, E)$, for $(\gamma, \eta) \in N_{(\alpha,\beta)} = N_\alpha \times N_\beta$. Likewise, the topology on $C^\alpha(U, C^\beta(V, E))$ is initial with respect to the inclusion maps

$$\Lambda_\gamma \colon C^\alpha(U, C^\beta(V, E)) \to C^\gamma(U, C^\beta(V, E)) \quad \text{for } \gamma \in N_\alpha.$$

Let $f \in C^{(\alpha,\beta)}(U \times V, E)$ and $x \in U$. Let $\gamma \in N_\alpha$. For each $\eta \in N_\beta$, the map f is $C^{(\gamma,\eta)}$, whence $f_x \in C^\eta(V, E)$, by Theorem A. Hence f_x is C^β. Since f is $C^{(\gamma,\eta)}$, the map $f^\vee$ is C^γ as a map to $C^\eta(V, E)$ (by Theorem A). Considering now $f^\vee$ as a map to $C^\beta(V, E)$, this means that $\iota_\eta \circ f^\vee$ is C^γ for each $\eta \in N_\beta$. Using Lemmas 7.4, 2.27 and 2.28, this implies that $f^\vee \colon U \to C^\beta(V, E)$ is C^γ. Hence, because $\gamma \in N_\alpha$ was arbitrary, $f^\vee \colon U \to C^\beta(V, E)$ is C^α.

It is clear that Φ is injective and linear. Let $\mathcal{T}$ be the initial topology on $C^\alpha(U, C^\beta(V, E))$ with respect to Φ. By the transitivity of initial topologies, this topology coincides with the initial topology with respect to the maps $\Lambda_\gamma \circ \Phi$, for $\gamma \in N_\alpha$. Again by transitivity, this topology coincides with the initial topology with respect to the maps $C^\gamma(U, \iota_\eta) \circ \Lambda_\gamma \circ \Phi$, for $\gamma \in N_\alpha$, $\eta \in N_\beta$. But $C^\gamma(U, \iota_\eta) \circ \Lambda_\gamma \circ \Phi = \Phi_{\gamma,\eta} \circ \lambda_{\gamma,\eta}$. Hence $\mathcal{T}$ coincides with the initial topology with respect to the maps $\Phi_{\gamma,\eta} \circ \lambda_{\gamma,\eta}$. By transitivity of initial topologies, this topology is initial with respect to the maps $\lambda_{\gamma,\eta}$ and hence coincides with the compact-open $C^{(\alpha,\beta)}$-topology on $C^{(\alpha,\beta)}(U \times V, E)$ Thus Φ is a linear topological embedding.

If $\mathbb{K}$ is metrizable and $g \in C^\alpha(U, C^\beta(V, E))$, define $g^\wedge \colon U \times V \to E$, $g^\wedge(x, y) :=$

$g(x)(y)$. Let $\gamma \in N_\alpha$ and $\eta \in N_\beta$. Then g is a C^γ-map to $C^\beta(V, E)$ and also (because the inclusion map ι_η is continuous linear) a C^γ-map to $C^\eta(V, E)$. Hence $g^\wedge = \Phi_{\gamma,\eta}^{-1}(g)$ is $C^{(\gamma,\eta)}$. Thus $g^\wedge$ is $C^{(\alpha,\beta)}$ and since $g = \Phi(g^\wedge)$, the linear embedding Φ is surjective and hence an isomorphism of topological vector spaces. $\square$

9 Further extensions and consequences

We collect consequences and generalizations of the previous results on Mahler expansions. Afterwards, we return to the approximation of functions by locally polynomial functions. We also obtain results concerning approximation by polynomial functions. Certain weighted c_0-spaces will be used:

Definition 9.1 Let X be a set, $\mathcal{W}$ be a non-empty set of functions $w\colon X \to]0, \infty[$ and E be a locally convex space over an ultrametric field $\mathbb{K}$. We set

$$c_0(X, \mathcal{W}, E) := \bigcap_{w \in \mathcal{W}} c_0(X, w, E)$$

and endow this space with the locally convex vector topology $\mathcal{O}$ defined by the set of ultrametric seminorms $\|.\|_{w,q}$, for $w \in \mathcal{W}$ and continuous ultrametric seminorms q on E. Thus $\mathcal{O}$ is the initial topology with respect to the inclusion maps

$$\Lambda_w\colon c_0(X, \mathcal{W}, E) \to c_0(X, w, E), \quad \text{for } w \in \mathcal{W}. \tag{36}$$

Proposition 9.2 *Let E be a sequentially complete locally convex p-adic vector space, $n_1, \ldots, n_\ell \in \mathbb{N}$, $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$, and N_α be the set of all $\beta = (\beta_1, \ldots, \beta_\ell) \in \mathbb{N}_0^{n_1} \times \cdots \times \mathbb{N}_0^{n_\ell} = \mathbb{N}_0^n$ such that $|\beta_j| \leq \alpha_j$ for all $j \in \{1, \ldots, \ell\}$. Let $f\colon (\mathbb{Z}_p)^n \to E$ be a continuous function and $a_\nu(f) \in E$ be the Mahler coefficients of f. Then f is C^α if and only if*

$$q(a_\nu(f))\nu^\beta \to 0 \quad \text{as } |\nu| \to \infty \tag{37}$$

for each $\beta \in N_\alpha$ and ultrametric continuous seminorm q on E. Moreover, the map

$$A_E^\alpha\colon C^\alpha(\mathbb{Z}_p^n, E) \to c_0(\mathbb{N}_0^n, \mathcal{W}_\alpha, E), \quad f \mapsto (a_\nu(f))_{\nu \in \mathbb{N}_0^n}$$

is an isomorphism of topological vector spaces, for $\mathcal{W}_\alpha := \{w_\beta\colon \beta \in N_\alpha\}$ with w_β as in 6.10.

Proof. The continuous map f is C^α if and only if it is C^β for each $\beta \in N_\alpha$ (see Definition 7.1). By Theorem A, this holds if and only if $(a_\nu(f))_{\nu\in\mathbb{N}_0^n} \in c_0(\mathbb{N}_0^n, w_\beta, E)$ for each $\beta \in N_\alpha$. By Definition 9.1, this condition is equivalent to $(a_\nu(f))_{\nu\in\mathbb{N}_0^n} \in c_0(\mathbb{N}_0^n, \mathcal{W}_\alpha, E)$. By Proposition 6.1, the map A_E^α is injective. To verify that A_E^α is surjective, let $b = (b_\nu)_{\nu\in\mathbb{N}_0^n} \in c_0(\mathbb{N}_0^n, \mathcal{W}_\alpha, E)$. Then $b \in c_0(\mathbb{N}_0^n, w_\beta, E)$ for each $\beta \in N_\alpha$. By Proposition 6.11, we find $f_\beta \in C^\beta(\mathbb{Z}_p^n, E)$ with Mahler coefficients b_ν. The uniqueness in Proposition 6.1 entails that $f := f_\beta$ is independent of β. Hence $f \in \bigcap_{\beta\in N_\alpha} C^\beta(\mathbb{Z}_p^n, E) = C^\alpha(\mathbb{Z}_p^n, E)$, and $A_E^\alpha(f) = b$ by construction. Hence A_E^α is surjective and hence an isomorphism of vector spaces. For $\beta \in N_\alpha$, let $\iota_\beta\colon C^\alpha(\mathbb{Z}_p^n, E) \to C^\beta(\mathbb{Z}_p^n, E)$ be the inclusion map. By Proposition 6.11, the topology on $C^\beta(\mathbb{Z}_p^n, E)$ is initial with respect to the map $A_E^\beta\colon C^\beta(\mathbb{Z}_p\, n, E) \to c_0(\mathbb{N}_0, w_\beta, E)$. Hence the compact-open C^α-topology $\mathcal{O}$ on $C^\alpha(\mathbb{Z}_p^n, E)$ is initial with respect to the maps $A_E^\beta \circ \iota_\beta$, for $\beta \in N_\alpha$. Let $\mathcal{T}$ be the topology on $C^\alpha(\mathbb{Z}_p^n, E)$ which is initial with respect to A_E^α. By transitivity of initial topologies, $\mathcal{T}$ is also initial with respect to the maps $\Lambda_{w_\beta} \circ A_E^\alpha = A_E^\beta \circ \iota_\beta$ (using notation as in (36)), and hence coincides with $\mathcal{O}$. $\square$

Corollary 9.3 *In the situation of Proposition 9.2, let $f\colon (\mathbb{Z}_p)^n \to E$ be a continuous function and $a_\nu(f) \in E$ be the Mahler coefficients of f. Let N'_α be the set of all $\beta = (\beta_1, \ldots, \beta_\ell) \in N_\alpha$ such that, for each $i \in \{1, \ldots, \ell\}$, the n_i-tupel $\beta_i \in \mathbb{N}_0^{n_i}$ has at most one non-zero component. Then f is C^α if and only if*

$$q(a_\nu(f))\nu^\beta \to 0 \quad \text{as } |\nu| \to \infty \tag{38}$$

for each $\beta \in N'_\alpha$ and ultrametric continuous seminorm q on E. Moreover, the map

$$C^\alpha(\mathbb{Z}_p^n, E) \to c_0(\mathbb{N}_0^n, \mathcal{W}'_\alpha, E), \quad f \mapsto (a_\nu(f))_{\nu\in\mathbb{N}_0^n}$$

is an isomorphism of topological vector spaces, for $\mathcal{W}'_\alpha := \{w_\beta\colon \beta \in N'_\alpha\}$.

Proof. Let J be the set of all $j = (j_1, \ldots, j_\ell)$ such that $j_i \in \{1, \ldots, n_i\}$ for $i \in \{1, \ldots, \ell\}$. If $\beta \in N_\alpha$ and $j \in J$, define $\beta[j] \in N_\alpha$ via $\beta[j] := (\eta_1, \ldots, \eta_\ell)$, where $\eta_i \in \mathbb{N}_0^{n_i}$ has $|\beta_i|$ as its j_i-th component, while all other components are 0. If $\nu = (\nu_1, \ldots, \nu_\ell) \in \mathbb{N}_0^{n_1} \times \cdots \times \mathbb{N}_0^{n_\ell}$ with $\nu_i = (\nu_{i,1}, \ldots, \nu_{i,n_i})$, then

$$\nu_i^{\beta_i} \leq \max\{\nu_{i,1}, \ldots, \nu_{i,n_i}\}^{|\beta_i|}$$

and hence

$$\nu^\beta \leq \max\{\nu_{1,j_i}^{|\beta_1|} \cdots \nu_{\ell,j_\ell}^{|\beta_\ell|}\colon (j_1, \ldots, j_\ell) \in J\}. \tag{39}$$

Hence, if (38) is satisfied for all $\beta \in N'_\alpha$, then also for all $\beta \in N_\alpha$. The converse is obvious as $N'_\alpha \subseteq N_\alpha$. Thus Proposition 9.2 shows that indeed f is C^α if and only if (38) is satisfied for all $\beta \in N'_\alpha$. By (39), we have

$$w_\beta \leq \max\{w_{\beta[j]} \colon j \in J\} \tag{40}$$

(as a pointwise maximum), entailing that $c_0(\mathbb{N}_0^n, \mathcal{W}_\alpha, E) = c_0(\mathbb{N}_0^n, \mathcal{W}'_\alpha, E)$ as a vector space. We also deduce from (40) that

$$\|.\|_{w_\beta,q} \leq \max\{\|.\|_{w_{\beta[j]},q} \colon j \in J\}$$

(as a pointwise maximum) for each ultrametric continuous seminorm q on E. Thus $c_0(\mathbb{N}_0^n, \mathcal{W}_\alpha, E) = c_0(\mathbb{N}_0^n, \mathcal{W}'_\alpha, E)$ as a topological vector space. The final assertion therefore follows from the final conclusion of Proposition 9.2. □

Remark 9.4 To deduce the final assertion of Theorem C from Corollary 9.3, note that

$$w_{re_i}(\nu) = \nu_i^r \leq |\nu|^r \leq n^r \max\{\nu_1^r, \ldots, \nu_n^r\} \leq n^r \max\{w_{re_i}(\nu) \colon i \in \{1, \ldots, n\}\}$$

for $\nu \in \mathbb{N}_0^n$.

Corollary 9.5 *Let E be a sequentially complete locally convex space over $\mathbb{Q}_p$ and $f \colon U \to E$ be a function on an open subset $U \subseteq \mathbb{Z}_p^n$. Let $k \in \mathbb{N}_0$. Then f is C^k if and only if f is $C^{(k,0,\ldots,0)}$, $C^{(0,k,0,\ldots,0)}$, ..., $C^{(0,\ldots,0,k,0)}$ and $C^{(0,\ldots,0,k)}$.*

Proof. Each point $x \in U$ has an open neighbourhood of the form $x + p^m\mathbb{Z}_p^n$ for some $m \in \mathbb{N}_0$. The assertions now follow from Corollary 9.3, applied (with $\ell = 1$ and $\alpha = (k)$) to the function $g \colon \mathbb{Z}_p^n \to E$, $g(y) := f(x + p^m y)$. □

Compare already [7, p. 140] for the characterization of C^1-functions $f \colon \mathbb{Z}_p^2 \to \mathbb{Q}_p$ via (38) (with $\ell = 1$). Related results for C^k-functions $\mathbb{Z}_p^n \to \mathbb{K}$ (where $\mathbb{K}$ is a finite field extension of $\mathbb{Q}_p$) and topologies on spaces of such functions can also be found in [20, 54–65].

The following lemma will help us to pass from compact cartesian sets to compact locally cartesian (and more general) sets in some results.

Lemma 9.6 *Let $(\mathbb{K}, |.|)$ be an ultrametric field, $n \in \mathbb{N}$, $U \subseteq \mathbb{K}^n$ be a compact locally cartesian set and $\mathcal{X}$ be an open cover of U. Then the following holds:*

(a) *There exist $m \in \mathbb{N}$ and disjoint compact, cartesian, relatively open subsets $W_1, \ldots, W_m$ of U subordinate to $\mathcal{X}$ such that $U = W_1 \cup \cdots \cup W_m$.*

(b) *The sets $W_1, \ldots, W_m$ in* (a) *can be chosen in such a way that $W_j = U \cap Q_j$ for certain disjoint clopen cartesian subsets $Q_1, \ldots, Q_m$ of $\mathbb{K}^n$.*

Proof. (a) Consider the ultrametric D on $\mathbb{K}^n$ defined via

$$D(x, y) := \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$$

for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{K}^n$. Let $d := D|_{U \times U}$ be the metric induced on U. Because U is locally cartesian, there exists a cover $\mathcal{V}$ of U consisting of cartesian, relatively open subsets $V \subseteq U$. Let $r > 0$ be a Lebesgue number for the open cover $\mathcal{V}$ of the compact metric space U. Thus, for each $x \in U$, there exists $V_x \in \mathcal{V}$ such that $B^d_r(x) \subseteq V_x$. Write $V_x = V_1 \times \cdots \times V_n$ with $V_1, \ldots, V_n \subseteq \mathbb{K}$. After decreasing r if necessary, we may assume that r also is a Lebesgue number for $\mathcal{X}$. Then

$$\begin{aligned} B^d_r(x) &= U \cap B^D_r(x) = V_x \cap B^D_r(x) = (V_1 \times \cdots \times V_n) \cap (B^{\mathbb{K}}_r(x_1) \times B^{\mathbb{K}}_r(x_n)) \\ &= (V_1 \cap B^{\mathbb{K}}_r(x_1)) \times \cdots \times (V_n \cap B^{\mathbb{K}}_r(x_n)) \end{aligned}$$

is a cartesian, relatively open subset of U. The ultrametric equality implies that, if $x, y \in U$, then either $B^d_r(x) = B^d_r(y)$ or $B^d_r(x) \cap B^d_r(y) = \emptyset$. Hence

$$P := \{B^d_r(x) \colon x \in U\}$$

is a partition of U into disjoint open and compact cartesian sets. By compactness of U, the set P is finite, say $P = \{U_1, \ldots, U_m\}$ with disjoint sets $U_1, \ldots, U_m$. Finally, because r is a Lebesgue number for $\mathcal{X}$, each $B^d_r(x)$ (and hence each U_j) is contained in some set $X \in \mathcal{X}$. Hence $U_1, \ldots, U_m$ is subordinate to $\mathcal{X}$.

(b) If $x, y \in U$ and $B^D_r(x) \cap B^D_r(y) \neq \emptyset$, then $B^D_r(x) = B^D_r(y)$ (as a consequence of the ultrametric inequality) and hence $B^d_r(x) = B^d_r(y)$. Therefore, if $U_j = B^d_r(x_j)$ in the proof of (a) (for $j \in \{1, \ldots, m\}$), then also the clopen subsets $Q_1 := B^D_r(x_1), \ldots, Q_m := B^D_r(x_m)$ of $\mathbb{K}^n$ are disjoint. By construction, $U_j = U \cap Q_j$. □

Lemma 9.7 *Let (X, d) be a complete metric space which is perfect (i.e., without isolated points). Then every compact subset K of X is contained in a perfect compact subset L of X.*

Proof. We may assume that $K \neq \emptyset$. Being compact and metrizable, K has a countable dense subset. Thus, we find $x_n \in K$ for $n \in \mathbb{N}$ such that $D := \{x_n : n \in \mathbb{N}\}$ is dense in K. Define $F_1 := \{x_1\}$. For $x \in F_1 \cup \{x_2\}$, choose $y_{2,x} \in X \setminus \{x\}$ such that $d(x, y_{2,x}) < 2^{-2}$, and set

$$F_2 := F_1 \cup \{x_2\} \cup \{y_{2,x} : x \in F_1 \cup \{x_2\}\}.$$

Recursively, if F_{n-1} has already been defined, choose $y_{n,x} \in X \setminus \{x\}$ for $x \in F_{n-1} \cup \{x_n\}$, such that $d(x, y_{n,x}) < 2^{-n}$, and set

$$F_n := F_{n-1} \cup \{x_n\} \cup \{y_{n,x} : x \in F_{n-1} \cup \{x_n\}\}.$$

Because $F := \bigcup_{n \in \mathbb{N}} F_n$ contains D, its closure $L := \overline{F}$ contains K as a subset. To see that L does not have isolated points, it suffices to show this for F. Now each $x \in F$ is an element of F_m for some $m \in \mathbb{N}$. Then $x \in F_n$ for each $n > m$, and F_n contains the point $y_{n,x} \neq x$ with $d(x, y_{n,x}) < 2^{-n}$. Hence x is not isolated in F.

Note that each set F_n is finite and $F_1 \subseteq F_2 \subseteq \cdots$. For each $m \in \mathbb{N}$, we have

$$F_{m+1} \subseteq B^d_{2^{-(m+1)}}(F_m) \cup B^d_{2^{-(m+1)}}(K) = B^d_{2^{-(m+1)}}(F_m \cup K)$$

and hence, by a simple induction based on the triangle inequality,

$$F_{m+n} \subseteq B^d_{2^{-(m+1)}+\cdots+2^{-(m+n)}}(F_m \cup K) \subseteq B^d_{2^{-m}}(F_m \cup K)$$

for each $n \in \mathbb{N}$. Therefore

$$F \subseteq B^d_{2^{-m}}(F_m \cup K). \tag{41}$$

Because L, being closed in X, is complete in the induced metric, it will be compact if we can show that it is precompact. To prove this, let $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that $2^{-m} < \frac{\varepsilon}{2}$. Because K is compact, there is a finite subset $\Phi \subseteq K$ such that $K \subseteq \bigcup_{x \in \Phi} B^d_{\varepsilon/2}(x)$. Then $\Phi \cup F_m$ is a finite subset of L and we claim that

$$L \subseteq \bigcup_{x \in \Phi \cup F_m} B^d_{\varepsilon}(x) \tag{42}$$

(whence L is precompact and the proof is complete). To prove the claim, it suffices to show that

$$F \subseteq \bigcup_{x \in \Phi \cup F_m} \overline{B}^d_{2^{-m}+\frac{\varepsilon}{2}}(x) \tag{43}$$

(as F is dense in L and the right-hand side is closed and contained in that of (42)). Let $y \in F$. By (41), we have $y \in B^d_{2^{-m}}(x)$ for some $x \in F_m \cup K$. If $x \in F_m$, then y is in the right-hand side of (43). If $x \in K$, then $x \in B^d_{\varepsilon/2}(z)$ for some $z \in \Phi$ and thus $y \in B^d_{2^{-m}}(x) \subseteq B^d_{2^{-m}+\frac{\varepsilon}{2}}(z)$, which is a subset of the union in (43). $\square$

Lemma 9.8 *Let $(\mathbb{K}, |.|)$ be a complete ultrametric field and $U \subseteq \mathbb{K}^n$ be a locally closed, locally cartesian subset. Then the following holds:*

(a) *If $x \in U$, then every neighbourhood $W \subseteq U$ of x contains an open cartesian neighbourhood $V \subseteq U$ of x which is closed in $\mathbb{K}^n$.*

(b) *If $V \subseteq U$ is an open cartesian subset which is closed in $\mathbb{K}^n$, then every compact subset $K \subseteq V$ is contained in a compact cartesian subset $L \subseteq V$.*

(c) *Every compact set $K \subseteq U$ is contained in a compact, locally cartesian subset of U.*

(d) *Consider U as a subset of $\mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_\ell}$ with $n_1 + \cdots + n_\ell = n$, and let $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$. Then the topology on $C^\alpha(U, E)$ is initial with respect to the restriction maps*

$$\rho_K \colon C^\alpha(U, E) \to C^\alpha(K, E),$$

for K ranging through the set of all compact cartesian subsets of U.

(e) *Every 0-neighbourhood in $C^\alpha(U, E)$ contains a 0-neighbourhood of the form $\rho_K^{-1}(Q)$ for some compact, locally cartesian subset $K \subseteq U$ and 0-neighbourhood $Q \subseteq C^\alpha(K, E)$.*

Proof. (a) Because U is locally cartesian, after shrinking W, we may assume that W is cartesian. Since U is locally closed, there exists a neighbourhood $C \subseteq U$ of x such that $C \subseteq W$. Let C^0 be the interior of C relative U. There exists a clopen cartesian subset $Q \subseteq \mathbb{K}^n$ with $x \in Q$ (e.g., a small ball around x with respect to the maximum norm) such that $U \cap Q \subseteq C^0$. Then $V := U \cap Q$ is open in U and $V = C \cap V = C \cap U \cap Q = C \cap Q$ is closed in $\mathbb{K}^n$. Moreover, $V = W \cap Q$ is cartesian (see 2.3).

(b) We have $V = V_1 \times \cdots \times V_n$ with closed subsets $V_1, \ldots, V_n \subseteq \mathbb{K}$ without isolated points. Being closed in $\mathbb{K}$, the sets V_j are complete. Let $\mathrm{pr}_j \colon \mathbb{K}^n \to \mathbb{K}$ be the projection onto the j-th component, for $j \in \{1, \ldots, n\}$. Then $K_j := \mathrm{pr}_j(K)$ is a compact subset of V_j and hence contained in a compact perfect subset $L_j \subseteq V_j$, by Lemma 9.7. Now $L := L_1 \times \cdots \times L_n$ is a compact, cartesian subset of V and contains K.

(c) Let $\mathcal{V}$ be the set of all open, cartesian subsets of U which are closed in $\mathbb{K}^n$. By (a), $\mathcal{V}$ is an open cover of U. Define the ultrametric D on $\mathbb{K}^n$ as in the proof of Lemma 9.6 (a). Let $d := D|_{U \times U}$ be the metric induced on U and $d_K := d|_{K \times K}$ be the metric induced on K. We let $\delta > 0$ be a Lebesgue number for the open cover $\mathcal{V}$ of K. Thus, for each $x \in K$, there exists $V_x \in \mathcal{V}$ such that $B^d_\delta(x) \subseteq V_x$. Because K is compact and d_K is an ultrametric, we have $\{B^{d_K}_\delta(x) \colon x \in K\} = \{W_1, \ldots, W_m\}$ with finitely many disjoint set $W_1, \ldots, W_m$. We have $W_j = B^{d_K}_\delta(x_j)$ for some $x_j \in K$. Then also the sets $Y_j := B^d_\delta(x_j)$ are disjoint for $j \in \{1, \ldots, m\}$. Note that $Y_j = B^D_\delta(x_j) \cap V_{x_j}$ is cartesian (see 2.3), open in U and closed in $\mathbb{K}^n$. The sets $K_j := K \cap Y_j$ being compact, (b) provides a compact cartesian subset $L_j \subseteq Y_j$ such that $K_j \subseteq L_j$. Because the cartesian sets L_j are disjoint and closed, it is clear that $L := L_1 \cup \cdots \cup L_m$ is locally cartesian. Moreover, L is compact, a subset of U, and contains $K = K_1 \cup \cdots \cup K_m$.

(d) Let $\mathcal{T}$ be the initial topology on $C^\alpha(U, E)$ with respect to the mappings ρ_K. By Remark 7.5 (a), the compact-open C^α-topology $\mathcal{O}$ on $C^\alpha(U, E)$ makes each of the maps ρ_K continuous and linear. Hence $\mathcal{T} \subseteq \mathcal{O}$. For the converse, let $W \subseteq C^\alpha(U, E)$ be a 0-neighbourhood. Let $\mathcal{V}$ be the set of all open cartesian subsets of U which are closed in $\mathbb{K}^n$. By (a), $\mathcal{V}$ is a cover of U. Thus Lemma 7.6 provides $\beta_1, \ldots, \beta_m \in N_\alpha$ and $V_1, \ldots, V_m \in \mathcal{V}$ with $\beta_j = (\beta_{j,1}, \ldots, \beta_{j,n})$ and $V_j = V_{j,1} \times \cdots \times V_{j,n}$ for $j \in \{1, \ldots, m\}$; compact subsets $K_j \subseteq V_j^{<\beta_j>}$ such that $K_j = K_{j,1}^{1+\beta_{j1}} \times \cdots \times K_{j,n}^{1+\beta_{j,n}}$ with compact subsets $K_{j,i} \subseteq \mathbb{K}$ for $i \in \{1, \ldots, n\}$; and open 0-neighbourhoods $Q_1, \ldots, Q_m \subseteq E$ such that

$$P := \bigcap_{j=1}^{m} \Delta^{-1}_{\beta_j, V_j}(\lfloor K_j, Q_j \rfloor) \subseteq W,$$

where $\Delta_{\beta_j, V_j} \colon C^\alpha(U, E) \to C((V_j)^{<\beta_j>}, E)$, $f \mapsto (f|_{V_j})^{<\beta_j>}$. Then $K_{j,i}$ is a compact subset of $V_{j,i}$ and hence $C_j := K_{j,1} \times \cdots \times K_{j,n}$ a compact subset of V_j. By (b), there exists a compact cartesian subset $L_j = L_{j,1} \times \cdots \times L_{j,n} \subseteq V_j$ such that $C_j \subseteq L_j$ and thus $K_{j,i} \subseteq L_{j,i}$ for all i. Consider the continuous linear maps $\delta_{\beta_j, L_j} \colon C^\alpha(L_j, E) \to C(L_j^{<\beta_j>}, E)$, $f \mapsto f^{<\beta_j>}$. Then

$P_j := \delta^{-1}_{\beta_j, L_j}(\lfloor K_j, Q_j \rfloor)$ is an open 0-neighbourhood in $C^\alpha(L_j, E)$ and $P = \bigcap_{j=1}^m \rho^{-1}_{L_j}(P_j)$ is open with respect to $\mathcal{T}$. Hence $\mathcal{O} \subseteq \mathcal{T}$ and $\mathcal{O} = \mathcal{T}$ follows.

(e) By (d), each 0-neighbourhood $W \subseteq C^\alpha(U, E)$ contains a 0-neighbourhood of the form $P := \rho^{-1}_{K_1}(Q_1) \cap \cdots \cap \rho^{-1}_{K_m}(Q_m)$ for some $m \in \mathbb{N}$, compact cartesian sets $K_j \subseteq U$ and 0-neighbourhoods $Q_j \subseteq C^\alpha(K_j, E)$, for $j \in \{1, \ldots, m\}$. By (c), $K_1 \cup \cdots \cup K_m$ is contained in a compact, locally cartesian subset $K \subseteq U$. Let $\rho_K \colon C^\alpha(U, E) \to C^\alpha(K, E)$ and $\rho_{K_j,K} \colon C^\alpha(K, E) \to C^\alpha(K_j, E)$ be the restriction maps, which are continuous linear by Remark 7.5 (a). Then $Q := \bigcap_{j=1}^m \rho^{-1}_{K_j,K}(Q_j)$ is a 0-neighbourhood in $C^\alpha(K, E)$ and $\rho_{K_j,K} \circ \rho_K = \rho_{K_j}$ entails that $P = \rho^{-1}_K(Q)$. □

9.9 Let $\mathbb{K}$ be a field, E be a $\mathbb{K}$-vector space, $n_1, \ldots, n_\ell \in \mathbb{N}$, $n := n_1 + \cdots + n_\ell$ and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$. A function $p \colon U \to E$ on a subset $U \subseteq \mathbb{K}^n$ is called a *polynomial function of multidegree* $\leq \alpha$ if there exist $a_\beta \in E$ for multi-indices $\beta = (\beta_1, \ldots, \beta_\ell) \in \mathbb{N}_0^{n_1} \times \cdots \times \mathbb{N}_0^{n_\ell} = \mathbb{N}_0^n$ with $|\beta_j| \leq \alpha_j$ for $j \in \{1, \ldots, \ell\}$, such that $a_\beta = 0$ for all but finitely many β and

$$p(x) = \sum_{\beta \leq \alpha} x^\beta a_\beta \quad \text{for all } \ x = (x_1, \ldots, x_n) \in U.$$

We write $\mathrm{Pol}_{\leq\alpha}(U, E)$ for the space of all such p. If $(\mathbb{K}, |.|)$ is a valued field, $U \subseteq \mathbb{K}^n$ a subset and E a topological $\mathbb{K}$-vector space, we say that a function $f \colon U \to E$ is *locally polynomial* of multidegree $\leq \alpha$ if each $x \in U$ has an open neighbourhood V in U such that $f|_V = p$ for some polynomial function $p \colon \mathbb{K}^n \supseteq V \to E$ of multidegree $\leq \alpha$. We write $\mathrm{LocPol}_{\leq\alpha}(U, E)$ for the space of all locally polynomial E-valued functions of multidegree $\leq \alpha$ on U.

Lemma 9.10 *Let $(\mathbb{K}, |.|)$ be an ultrametric field, $U \subseteq \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_\ell} = \mathbb{K}^n$ be a compact locally cartesian set, where $n_1, \ldots n_\ell \in \mathbb{N}$ and $n = n_1 + \cdots + n_\ell$. Let $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$ and $f \in \mathrm{LocPol}_{\leq\alpha}(U, E)$. Then there exists a function $g \in \mathrm{LocPol}_{\leq\alpha}(\mathbb{K}^n, E)$ such that $g|_U = f$.*

Proof. We may assume that $U \neq \emptyset$. Let $\mathcal{X}$ be the set of all open subsets $X \subseteq U$ such that $f|_X \in \mathrm{Pol}_{\leq\alpha}(X, E)$. By Lemma 9.6, there exist disjoint clopen subsets $Q_1, \ldots, Q_m$ of $\mathbb{K}^n$ such that the intersections $W_j := Q_j \cap U$ form an open cover of U for $j \in \{1, \ldots, m\}$ and $W_j \subseteq X_j$ for some $X_j \in \mathcal{X}$. Thus, there are $p_j \in \mathrm{Pol}_{\leq\alpha}(\mathbb{K}^n, E)$ such that $p_j|_{W_j} = f|_{W_j}$. Define $g \colon \mathbb{K}^n \to E$ via $g(x) := p_j(x)$ if $x \in Q_j$, $g(x) := 0$ if $x \in \mathbb{K}^n \setminus (Q_1 \cup \cdots \cup Q_m)$. Because

the sets Q_m are clopen and disjoint, we have $g \in \mathrm{LocPol}_{\leq\alpha}(\mathbb{K}^n, E)$. Also $g|_U = f$, using that $g|_{W_j} = p_j|_{W_j} = f|_{W_j}$ and $U = W_1 \cup \cdots \cup W_m$. $\square$

Then the following analogue of Theorem B holds:

Proposition 9.11 *For every complete ultrametric field $\mathbb{K}$, locally convex topological $\mathbb{K}$-vector space E, $n_1, \ldots, n_\ell \in \mathbb{N}$, $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$ and locally closed, locally cartesian subset $U \subseteq \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_\ell} = \mathbb{K}^n$ with $n := n_1 + \cdots + n_\ell$, the space $\mathrm{LocPol}_{\leq\alpha}(U, E)$ of E-valued locally polynomial functions of multidegree $\leq \alpha$ is dense in $C^\alpha(U, E)$.*

Proof. Step 1: If U is a compact cartesian subset of $\mathbb{K}^n$, we can prove the assertion by induction on $n \in \mathbb{N}$:

The case $n = 1$. Then $\alpha = (r)$ with $r := \alpha_1$. For $E = \mathbb{K}$ and $r < \infty$, the assertion was established in [20, Proposition II.40], and one can proceed analogously if $r < \infty$ and E is an ultrametric Banach space. We can now pass to general locally convex spaces and $r \in \mathbb{N}_0 \cup \{\infty\}$ as in the proof of Theorem B, using the analogues of Lemmas 3.8, 3.12 (subsuming 3.14), 3.16 and 3.17 for C^α-maps on $U \subseteq \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_\ell}$ described in Remark 7.5 (c), (d), (f) and (g), respectively.

Induction step. If $n \geq 2$, we write $U = U_1 \times U'$ with compact cartesian subsets $U_1 \subseteq \mathbb{K}^{n_1}$ and $U' \subseteq \mathbb{K}^{n_2} \times \cdots \times \mathbb{K}^{n_\ell}$. We also write $\alpha = (r, \alpha')$ with $r \in \mathbb{N}_0 \cup \{\infty\}$ and $\alpha' \in (\mathbb{N}_0 \cup \{\infty\})^{\ell-1}$. Using Theorem D instead of Theorem A and replacing $\sum_{i=0}^r$ with $\sum_{|\nu|\leq r}$ and i with ν (where the summation is over all $\nu \in \mathbb{N}_0^{n_1}$ such that $|\nu| \leq r$), we can perform the induction step as in the proof of Theorem B.

Step 2: If U is a compact, locally cartesian subset of $\mathbb{K}^n$, then $U = W_1 \cup \cdots \cup W_m$ is a disjoint union of certain compact, relatively open, cartesian subsets $W_1, \ldots, W_m$ of U (see Lemma 9.6). As a consequence of Lemma 2.24 and Remark 7.5 (b), the map

$$\rho = (\rho_j)_{j=1}^m \colon C^\alpha(U, E) \to \prod_{j=1}^m C^\alpha(W_j, E), \quad f \mapsto (f|_{W_j})_{j=1}^m$$

is an isomorphism of topological vector spaces. Let $f \in C^\alpha(U, E)$ and $P \subseteq C^\alpha(U, E)$ be a neighbourhood of f. By the preceding, we may assume that $P = \rho^{-1}(P_1 \times \cdots \times P_m)$ with neighbourhoods $P_j \subseteq C^\alpha(W_j, E)$

of $f|_{W_j}$ for $j \in \{1,\ldots,m\}$. By Step 1, for each j there exists some $g_j \in P_j \cap \mathrm{LocPol}_{\leq\alpha}(W_j,E)$. We define $g\colon U \to E$ via $g(x) := g_j(x)$ if $x \in W_j$. Then $g \in P$ and $g \in \mathrm{LocPol}_{\leq\alpha}(U,E)$, showing that the latter is dense.

Step 3. Finally, let $U \subseteq \mathbb{K}^n$ be a locally closed, locally cartesian subset, $f \in C^\alpha(U,E)$ and $P \subseteq C^\alpha(U,E)$ be a 0-neighbourhood. By Lemma 9.8 (e), we may assume that

$$P = \rho_K^{-1}(Q)$$

for some compact, locally cartesian subset K of U and open 0-neighbourhood $Q \subseteq C^\alpha(K,E)$. By Step 2, there exists $g \in (f|_K + Q) \cap \mathrm{LocPol}_{\leq\alpha}(K,E)$. Using Lemma 9.10, we find $h \in \mathrm{LocPol}_{\leq\alpha}(U,E)$ such that $h|_K = g$. Then $h \in \rho_K^{-1}(f|_K + Q) = f + P$ and thus $\mathrm{LocPol}_{\leq\alpha}(U,E)$ is dense. □

Proposition 9.12 *For every complete ultrametric field $\mathbb{K}$, locally convex topological $\mathbb{K}$-vector space E, $n_1,\ldots,n_\ell \in \mathbb{N}$, $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$ and locally closed, locally cartesian subset $U \subseteq \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_\ell} = \mathbb{K}^n$ with $n := n_1 + \cdots + n_\ell$, the space $\mathrm{Pol}(U,E)$ of E-valued polynomial functions is dense in $C^\alpha(U,E)$.*

Proof. Step 1. If $U \subseteq \mathbb{K}^n$ is a compact cartesian set, the proof is by induction on ℓ.

If $\ell = 1$, then $\alpha = (r)$ with $r := \alpha_1$. For $E = \mathbb{K}$ and $r < \infty$, the assertion was established in [20, Corollary II.42], and one can proceed analogously if $r < \infty$ and E is an ultrametric Banach space.

Now let E be a locally convex space, $f \in C^r(U,E)$ and $W \subseteq C^r(U,E)$ be an open 0-neighbourhood.

The case $r < \infty$. Let $\widetilde{E}$ and $\phi_q\colon \widetilde{E} \to E_q$ be as in the proof of Theorem B. Repeating the arguments (with Remark 7.5 (d) and (f) instead of Lemmas 3.14 and 3.16), we may assume that $W = C^r(U,E) \cap \widetilde{W}$ for some open 0-neighbourhood $\widetilde{W} \subseteq C^r(U,\widetilde{E})$, and $\widetilde{W} = C^r(U,\phi_q)^{-1}(V)$ for some q and open 0-neighbourhood $V \subseteq C^r(U,E_q)$. By the case of ultrametric Banach spaces, there exists $g \in \mathrm{Pol}(U,E_q)$ such that $\phi_q \circ f - g \in V$. Thus, there exists a finite subset $\Theta \subseteq \mathbb{N}_0^n$ and $c_\nu \in E_q$ for $\nu \in \Theta$ such that

$$g(x) = \sum_{\nu\in\Theta} x^\nu c_\nu.$$

By Lemma 3.17, there are open neighbourhoods $Q_\nu \subseteq E_q$ of c_ν such that

$$\phi_q \circ f - \sum_{\nu \in \Theta} x^\nu b_\nu \in V,$$

for all $b_\nu \in Q_\nu$. Since $\phi_q(E)$ is dense in E_q, there exist $a_\nu \in E$ such that $\phi_q(a_\nu) \in Q_\nu$. Hence $h := \sum_{\nu \in \Theta} x^\nu a_\nu \in \mathrm{Pol}(U, E)$ and

$$\phi_q \circ (f - h) = \phi_q \circ f - \sum_{\nu \in \Theta} x^\nu a_\nu \in V.$$

Thus $f - h \in C^r(U, \phi_q)^{-1}(V) = \widetilde{W}$ and thus $f - h \in \widetilde{W} \cap E = W$.

If $r = \infty$, we can argue as in the proof of Theorem B, replacing Lemma 3.8 with Remark 7.5 (c).

Induction step. If $\ell \geq 2$, we write $U = U_1 \times U'$ with compact cartesian sets $U \subseteq \mathbb{K}^{n_1}$ and $U' \subseteq \mathbb{K}^{n_2} \times \cdots \times \mathbb{K}^{n_\ell}$. We also write $\alpha = (r, \alpha')$ with $r \in \mathbb{N}_0 \cup \{\infty\}$ and $\alpha' \in (\mathbb{N}_0 \cup \{\infty\})^{\ell - 1}$. Let $f \in C^\alpha(U, E)$ and $W \subseteq C^\alpha(U, E)$ be an open 0-neighbourhood. By Theorem D, the map

$$\Phi \colon C^\alpha(U, E) \to C^r(U_1, C^{\alpha'}(U', E)), \quad f \mapsto f^\vee$$

is an isomorphism of topological vector spaces. Hence $W = \Phi^{-1}(V)$ for some open 0-neighbourhood $V \subseteq C^r(U_1, C^{\alpha'}(U', E))$. By induction, there exists $g \in \mathrm{Pol}(U_1, C^{\alpha'}(U', E))$ with $\Phi(f) - g \in V$. Thus, there is a finite subset $\Theta \subseteq \mathbb{N}_0^{n_1}$ and $c_\nu \in C^{\alpha'}(U', E)$ for $\nu \in \Theta$ such that

$$g(x) = \sum_{\nu \in \Theta} x^\nu c_\nu.$$

By Remark 7.5 (g), there are open neighbourhoods $Q_\nu \subseteq C^{\alpha'}(U', E)$ of c_ν such that

$$\Phi(f) - \sum_{\nu \in \Theta} X^\nu b_\nu \in V$$

for all $b_\nu \in Q_\nu$, where $X^\nu \colon U_1 \to \mathbb{K}$, $x \mapsto x^\nu$. By induction, we find $b_\nu \in Q_\nu \cap \mathrm{Pol}(U', E)$. Then

$$g \colon U \to E, \quad g(x, x') := \sum_{\nu \in \Theta} x^\nu b_\nu(x') \quad \text{for } x \in U_1,\ x' \in U'$$

is an element of $\mathrm{Pol}(U,E)$. Since $\Phi(g)=\sum_{\nu\in\Theta} x^\nu b_\nu$, we have $\Phi(f)-\Phi(g)\in V$ and thus $f-g\in W$.

Step 2. Now assume that $U\subseteq\mathbb{K}^n$ is compact and locally cartesian. Given $f\in C^\alpha(U,E)$ and an open neighbourhood $W\subseteq C^\alpha(U,E)$ of f, we want to find $h\in W\cap\mathrm{Pol}(U,E)$. By Proposition 9.11, there exists $g\in W\cap\mathrm{LocPol}_{\leq\alpha}(U,E)$. After replacing f with g, we may assume that f is locally polynomial. Let $\mathcal{V}$ be a cover of U by open subsets $V\subseteq U$ on which f is polynomial. By Lemma 9.6, we can write $U=W_1\cup\cdots\cup W_m$ as a disjoint union of relatively open, compact, cartesian subsets $W_1,\ldots,W_m\subseteq U$, of the form $W_j=U\cap Q_j$ with clopen, cartesian, disjoint subsets $Q_1,\ldots,Q_m\subseteq\mathbb{K}^n$, such that each W_j is a subset of some $V_j\in\mathcal{V}$. By the last property, for each j we find $p_j\in\mathrm{Pol}(\mathbb{K}^n,E)$ such that $f|_{W_j}=p_j|_{W_j}$. Let $\mathrm{pr}_i\colon\mathbb{K}^n\to\mathbb{K}$ be the projection onto the i-th component, for $i\in\{1,\ldots,n\}$. Then $K_i:=\mathrm{pr}_i(U)\subseteq\mathbb{K}$ is compact and does not have isolated points (as $K_i=\bigcup_{j=1}^m\mathrm{pr}_i(W_j)$). Thus

$$K:=K_1\times\cdots\times K_n\subseteq\mathbb{K}^n$$

is a compact cartesian set. We define a (new) function $g\colon K\to E$ via $g(x):=p_j(x)$ if $x\in K\cap Q_j$, $g(x):=0$ if $x\in K\setminus(Q_1\cup\cdots\cup Q_m)$. Lemma 2.24 implies that $g\in C^\alpha(K,E)$. Moreover, $g|_U=f$. The restriction map $\rho\colon C^\alpha(K,E)\to C^\alpha(U,E)$ is continuous (Remark 7.5 (a)), whence $\rho^{-1}(W)$ is an open neighbourhood of g. By Step 1, there exists $h\in\rho^{-1}(W)\cap\mathrm{Pol}(K,E)$. Then $h|_U\in W\cap\mathrm{Pol}(U,E)$ and hence $\mathrm{Pol}(U,E)$ is dense.

Step 3. Now consider a locally closed, locally cartesian subset $U\subseteq\mathbb{K}^n$. Let $f\in C^\alpha(U,E)$ and $P\subseteq C^\alpha(U,E)$ be an open 0-neighbourhood. As in Step 3 of the proof of Proposition 9.11, we may assume that $P=\rho_K^{-1}(Q)$ for some open 0-neighbourhood $Q\subseteq C^\alpha(K,E)$ and compact, locally cartesian subset $K\subseteq U$. By Step 2 and Lemma 9.10, there exists $p\in\mathrm{Pol}(U,E)$ such that $f|_K-p|_K\in Q$. Thus $f-p\in\rho_K^{-1}(Q)=P$, and thus $\mathrm{Pol}(U,E)$ is dense. $\square$

A Outlook on related concepts and results

In this appendix, we give an alternative definition of C^α-maps, which applies to maps on subsets of arbitrary (possibly infinite-dimensional!) topological vector spaces. We also describe various assertions (to be proved elsewhere) which will make the theory of C^α-maps more complete (but were irrelevant

for our current ends). It is natural to approach them with the methods developed in [2], [3], [5], [11], [13], and [14].

We recall from [14] (and the earlier work [5], in the case of open sets):

Definition A.1 If $\mathbb{K}$ is a topological field, E a topological $\mathbb{K}$-vector space and $U \subseteq E$ a subset with dense interior, we define $U^{[0]} := U$ and

$$U^{[1]} := \{(x, y, t) \in U \times E \times \mathbb{K} \colon x + ty \in U\}\,.$$

Recursively, we let $U^{[k]} := (U^{[1]})^{[k-1]}$ for $k \in \mathbb{N}$. If also F is a topological $\mathbb{K}$-vector space and $f \colon U \to F$ a function, we say that f is C^0_{BGN} it f is continuous, and define $f^{[0]} := f$ in the case. We say that f is C^1_{BGN} if f is continuous and there exists a continuous map

$$f^{[1]} \colon U^{[1]} \to F$$

such that $f^{[1]}(x, y, t) = \frac{1}{t}(f(x + ty) - f(x))$ for all $(x, y, t) \in U^{[1]}$ with $t \neq 0$. Recursively, having defined when f is C^j_{BGN} and associated maps $f^{[j]}$ for $j \in \{0, 1, \ldots, k-1\}$, where $k \in \mathbb{N}$, we say that f is C^k_{BGN} if f is C^1_{BGN} and $f^{[1]}$ is C^{k-1}_{BGN}. Let $f^{[k]} := (f^{[1]})^{[k-1]} \colon U^{[k]} \to F$ in this case.

We now define the analogous C^α-maps.

Definition A.2 Let $\mathbb{K}$ be a topological field, $E_1, \ldots, E_\ell$ be topological $\mathbb{K}$-vector spaces and $U_j \subseteq E_j$ be subsets with dense interior, for $j \in \{1, \ldots, \ell\}$. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in (\mathbb{N}_0 \cup \{\infty\})^\ell$ and F be a topological $\mathbb{K}$-vector space. We say that a map $f \colon U_1 \times \cdots \times U_\ell \to F$ is C^α_{BGN} if there exist continuous mappings

$$f^{[\beta]} \colon U_1^{[\beta_1]} \times \cdots \times U_\ell^{[\beta_\ell]} \to F$$

for all $\beta \in \mathbb{N}_0^\ell$ with $\beta \leq \alpha$, such that

$$f^{[0,\beta_\ell]}(x, y) := (f(x, \bullet))^{[\beta_\ell]}(y)$$

for all $x \in U_1 \times \cdots \times U_{\ell-1}$ and $y \in U_\ell^{[\beta_\ell]}$ and, recursively,

$$f^{[0,\beta_j,\beta')]}(x, y, z) := (f^{[0,0,\beta']}(x, \bullet, z))^{[\beta_j]}(y)$$

for all $x \in U_1 \times \cdots \times U_{j-1}$, $y \in U_j^{[\beta_j]}$ and $z \in U_{j+1}^{[\beta_{j+1}]} \times \cdots \times U_\ell^{[\beta_\ell]}$, with $\beta' := (\beta_{j+1}, \ldots, \beta_\ell)$. Endow the space $C^\alpha(U, F)_{BGN}$ of all C^α_{BGN}-maps $f \colon U \to F$ with the initial topology with respect to the maps

$$C^\alpha(U, F)_{BGN} \to C(U^{[\beta]}, F), \quad f \mapsto f^{[\beta]}$$

(using the compact-open topology on the right-hand side), for $\beta \in \mathbb{N}_0^\ell$ such that $\beta \leq \alpha$.

Then the following holds:

A.3 For $E_1, \ldots, E_\ell$, F and $U_1, \ldots, U_\ell$ as before and $r \in \mathbb{N}_0 \cup \{\infty\}$, a map $f \colon U_1 \times \cdots \times U_\ell \to F$ is C^r_{BGN} if and only if f is C^α for all $\alpha \in \mathbb{N}_0^\ell$ such that $|\alpha| \leq r$ (cf. [5, Lemma 3.9] to get an induction started). More generally, if also $H_1, \ldots, H_k$ are topological $\mathbb{K}$-vector spaces, $V_i \subseteq H_i$ subsets with dense interior for $i \in \{1, \ldots, k\}$, $\beta \in (\mathbb{N}_0 \cup \{\infty\})^k$ and

$$f \colon V_1 \times \cdots \times V_k \times U_1 \times \cdots \times U_\ell \to F$$

a map, then f is $C^{(\beta,\alpha)}$ for all α as before if and only if f is $C^{(\beta,r)}$ on $V_1 \times \cdots \times V_k \times U \subseteq H_1 \times \cdots \times H_k \times E$, with $E := E_1 \times \cdots \times E_\ell$ and $U := U_1 \times \cdots \times U_\ell$.

A.4 Consider $E_j = \mathbb{K}^{n_j}$ for $j \in \{1, \ldots, \ell\}$, locally cartesian subsets $U_j \subseteq E_j$ with dense interior, and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$. Let $U := U_1 \times \cdots \times U_\ell$ and $f \colon U \to F$ be a function from U to a topological $\mathbb{K}$-vector space F. Then:

(a) f is C^α in the sense of Definition 7.1 if and only if f is C^α_{BGN}. Moreover, $C^\alpha(U, F) = C^\alpha(U, F)_{BGN}$ as a topological vector space.[12]

(b) If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and F is locally convex, then f is C^α if and only if the partial derivatives $\partial^\beta f$ exist on the interior of U and extend to continuous functions $\partial^\beta f \colon U \to F$ (denoted by the same symbol), for all $\beta \in N_\alpha$.

See [2], [3, Lemmas 3.17 and 3.18], [5, Proposition 4.5], [14, 1.10] and [20, Proposition 2.18] for special cases of the following Chain Rule:

A.5 If $f \colon U_1 \times \cdots \times U_\ell \to F$ is C^α_{BGN} and $g_j \colon V_{j,1} \times \cdots \times V_{j,k_j} \to U_j \subseteq E_j$ is C^{β_j} (resp., $C^{\beta_j}_{BGN}$) and $|\beta_j| \leq \alpha_j$ for $j \in \{1, \ldots, \ell\}$, then the map

$$f \circ (g_1 \times \cdots \times g_\ell) \colon (V_{1,1} \times \cdots \times V_{1,k_1}) \times \cdots \times (V_{\ell,1} \times \cdots \times V_{\ell,k_\ell}) \to F$$

is $C^{(\beta_1,\ldots,\beta_\ell)}$ (resp., $C^{(\beta_1,\ldots,\beta_\ell)}_{BGN}$). The composition is also $C^{(\beta_1,\ldots,\beta_\ell)}$ if f is C^α and each g_j is C^{β_j}.

[12] See already [11, Theorem A] for the case of C^r-functions, and Theorem B.2 of its preprint-version `arXiv:math/0609041v1` for the equality of the C^r-topologies.

A.6 If $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, all of $E_1, \ldots, E_\ell, F$ are locally convex and $U_1, \ldots, U_\ell$ open, then the C^α_{BGN}-maps of Definition A.2 should also coincide with the C^α-maps discussed in [3] (in the case $\ell = 2$) and the work in progress [2] (see [5, Proposition 7.4] for the case of C^r-maps). Moreover, both approaches should give the same topology on $C^\alpha(U, E)$ (as in the case of C^r-maps settled in [13, Proposition 4.19 (d)]).

A.7 If $E_1, \ldots, E_\ell$, $U_1 \ldots, U_\ell$ and α are as in Definition A.2, $H := H_1 \times \cdots \times H_k$ a direct product of topological $\mathbb{K}$-vector spaces and $f\colon U_1 \times \cdots \times U_\ell \times H \to F$ a $C^{(\alpha,0)}$-map such that $f(x, \bullet)\colon H_1 \times \cdots \times H_k \to F$ is k-linear for all $x \in U_1 \times \cdots \times U_\ell$, then f is $C^{(\alpha,\infty)}$ (cf. [3, Lemma 3.14] for a special case).

A.8 Assume that $E_1, \ldots, E_\ell$, $F_1 \ldots, F_k$ and H are topological vector spaces over the topological field $\mathbb{K}$, $U \subseteq E_1 \times \cdots \times E_\ell =: E$ and $V \subseteq F_1 \times \cdots \times F_\ell =: F$ subsets with dense interior. Let $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$ and $\beta \in (\mathbb{N}_0 \cup \{\infty\})^k$. Then $f^\vee(x) := f(x, \bullet) \in C^\beta(V, H)$ for each $f \in C^{(\alpha,\beta)}(U \times V, H)$ and $x \in U$, the map $f^\vee\colon U \to C^\beta(V, H)$ is C^α, and

$$\Phi\colon C^{(\alpha,\beta)}(U \times V, H) \to C^\alpha(U, C^\beta(V, H)), \quad f \mapsto f^\vee$$

is a linear topological embedding. If $\mathbb{K}$, E and F are metrizable or $\mathbb{K}$ and V are locally compact, then Φ is a topological embedding.[13]

A.9 If $\mathbb{K}$ and $U = U_1 \times \cdots \times U_\ell$ is locally compact in the situation of Definition A.2, then the evaluation map

$$C^\alpha(U, F) \times U_1 \times \cdots \times U_\ell \to F, \quad (f, x) \mapsto f(x)$$

is $C^{(\infty,\alpha)}$.

(See [2], [3, Proposition 3.20] and [13, Proposition 11.1] for special cases, and combine them with A.7 to increase the order of differentiability in the linear first argument). Also the differentiability properties of the composition $f \circ g$ as a function of (f, g) can be analyzed (generalizing the discussions in [2] and [13, Proposition 11.2]).

A.10 If $U_j \subseteq \mathbb{K}^{n_j}$ is locally cartesian for $j \in \{1, \ldots, \ell\}$, $U := U_1 \times \cdots \times U_\ell \subseteq \mathbb{K}^n$ with $n := n_1 + \cdots + n_\ell$, $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^\ell$ and $f\colon U \to E$ is a C^α-function as in Definition 7.1, then $f^{<\beta>}\colon U_1^{<\beta_1>} \times \cdots \times U_\ell^{<\beta_\ell>} \to F$ is a $C^{<\alpha - (|\beta_1|, \ldots, |\beta_\ell|)>}$-function, for each $\beta = (\beta_1, \ldots, \beta_\ell) \in N_\alpha$.

[13] Less is needed: Assume that $V^{[\eta]}$ is locally compact for each $\eta \in \mathbb{N}_0^k$ such that $\eta \leq \beta$, or that $(U \times V)^{[\gamma]}$ is a k-space for all $\gamma \in \mathbb{N}_0^{\ell+k}$ such that $\gamma \leq (\alpha, \beta)$.

B The compact-open topology

In this appendix, we give a self-contained introduction to the compact-open topology on function spaces. Most of the results are classical, and no originality is claimed (cf. [6, Chapter X, §1–§3] and [8], for example). However, the sources treating the topic do so with a different thrust. Here, we compile precisely those results which are the foundation for the study of non-linear mappings between function spaces, and their differentiability properties.

We recall: If X and Y are Hausdorff topological spaces, then the compact-open topology on $C(X,Y)$ is the topology given by the subbasis of open sets

$$\lfloor K,U\rfloor := \{\gamma \in C(X,Y)\colon \gamma(K)\subseteq U\}, \tag{44}$$

for K ranging through the set $\mathcal{K}(X)$ of all compact subsets of X and U though the set of open subsets of Y. In other words, finite intersections of sets as in (44) form a basis for the compact-open topology on $C(X,Y)$. We always endow $C(X,Y)$ with the compact-open topology (unless the contrary is stated).

Remark B.1 The compact-open topology on $C(X,Y)$ makes the point evaluation

$$\mathrm{ev}_x\colon C(X,Y)\to Y,\quad \gamma\mapsto\gamma(x)$$

continuous, for each $x\in X$. (If $U\subseteq Y$ is open, then $\mathrm{ev}_x^{-1}(U)=\lfloor\{x\},U\rfloor$ is open in $C(X,Y)$). As the maps ev_x separate points on $C(X,Y)$ for $x\in X$, it follows that the compact-open topology on $C(X,Y)$ is Hausdorff.

Lemma B.2 *Let X and Y be Hausdorff topological spaces and $\mathcal{S}$ be a subbasis of open subsets of Y. Then*

$$\mathcal{V} := \{\lfloor K,U\rfloor\colon K\in\mathcal{K}(X), U\in\mathcal{S}\}$$

is a subbasis for the compact-open topology on $C(X,Y)$.

Proof. Each of the sets $V\in\mathcal{V}$ is open in the compact-open topology. To see that $\mathcal{V}$ is a subbasis for the latter, it suffices to show that $\lfloor K,U\rfloor$ is open in the topology generated by $\mathcal{V}$, for each $K\in\mathcal{K}(X)$ and open subset $U\subseteq Y$. To check this, let $\gamma\in\lfloor K,U\rfloor$. For each $x\in K$, there is a finite subset $F_x\subseteq\mathcal{S}$ such that

$$\gamma(x)\in\bigcap_{S\in F_x} S\subseteq U.$$

Set $S_x := \bigcap_{S \in F_x} S$. Since K is compact and hence locally compact, each $x \in K$ has a compact neighbourhood $K_x \subseteq \gamma^{-1}(S_x)$ in K. By compactness, there is a finite subset $\Phi \subseteq K$ such that $K = \bigcup_{x \in \Phi} K_x^0$ (where K_x^0 denotes the interior of K_x relative K). Then $V := \bigcap_{x \in \Phi} \lfloor K_x, S_x \rfloor = \bigcap_{x \in \Phi} \bigcap_{S \in F_x} \lfloor K_x, S \rfloor$ is open in the topology generated by $\mathcal{V}$, and $V \subseteq \lfloor K, U \rfloor$. In fact, let $\gamma \in V$. For each $y \in K$, there is $x \in \Phi$ such that $y \in K_x$. Hence $\gamma(y) \in \gamma(K_x) \subseteq S_x \subseteq U$ and thus $\gamma \in \lfloor K, U \rfloor$. $\square$

Mappings between function spaces of the form $C(X, f)$ (so-called superposition operators) are used frequently.

Lemma B.3 *Let X, Y_1 and Y_2 be Hausdorff topological spaces. If a map $f\colon Y_1 \to Y_2$ is continuous, then also the following map is continuous:*

$$C(X, f)\colon C(X, Y_1) \to C(X, Y_2), \quad \gamma \mapsto f \circ \gamma\,.$$

Proof. The map $C(X, f)$ will be continuous if pre-images of subbasic open sets are open. To this end, let $K \in \mathcal{K}(X)$ and $U \subseteq Y_2$ be open. Then $C(X, f)^{-1}(\lfloor K, U \rfloor) = \lfloor K, f^{-1}(U) \rfloor$ is open in $C(X, Y_1)$ indeed. $\square$

Lemma B.4 *Let X and Y be Hausdorff topological spaces. Assume that the topology on Y is initial with respect to a family $(f_j)_{j \in J}$ of maps $f_j\colon Y \to Y_j$ to Hausdorff topological spaces Y_j. Then the compact-open topology on $C(X, Y)$ is initial with respect to the family $(C(X, f_j))_{j \in J}$ of the mappings $C(X, f_j)\colon C(X, Y) \to C(X, Y_j)$.*

Proof. Let $\mathcal{S}$ be the set of all subsets of Y of the form $f_j^{-1}(W)$, with $j \in J$ and W an open subset of Y_j. By hypothesis, $\mathcal{S}$ is a subbasis for the topology of Y. Hence, by Lemma B.2, the sets $\lfloor K, f_j^{-1}(W) \rfloor$ form a subbasis for the compact-open topology on $C(X, Y)$, for j, W as before and $K \in \mathcal{K}(X)$. But $\lfloor K, f_j^{-1}(W) \rfloor = C(X, f_j)^{-1}(\lfloor K, W \rfloor)$ (since $\gamma(K) \subseteq f_j^{-1}(W)$ $\Leftrightarrow$ $f_j(\gamma(K)) \subseteq W \Leftrightarrow (f_j \circ \gamma)(K) \subseteq W$), and these sets form a subbasis for the initial topology on $C(X, Y)$ with respect to the family $(C(X, f_j))_{j \in J}$. $\square$

We mention three consequences of Lemma B.4.

Lemma B.5 *If X is a Hausdorff topological space and $f\colon Y_1 \to Y_2$ is a topological embedding of Hausdorff topological spaces, then also the mapping $C(X, f)\colon C(X, Y_1) \to C(X, Y_2)$ is a topological embedding.*

Proof. Since f is injective, also $C(X, f)$ is injective. By hypothesis, the topology on Y_1 is initial with respect to f. Hence the topology on $C(X, Y_1)$ is initial with respect to $C(X, f)$ (by Lemma B.4). As a consequence, the injective map $C(X, f)$ is a topological embedding. $\square$

Remark B.6 In particular, if X and Y_2 are Hausdorff topological spaces and $Y_1 \subseteq Y_2$ is a subset, endowed with the induced topology, then topology induced by $C(X, Y_2)$ on $C(X, Y_1)$ coincides with the compact-open topology on $C(X, Y_1)$.

Next, we deduce that $C(X, \prod_{j\in J} Y_j) = \prod_{j\in J} C(X, Y_j)$.

Lemma B.7 *Let X be a Hausdorff topological space and $(Y_j)_{j\in J}$ be a family of Hausdorff topological spaces, with cartesian product $Y := \prod_{j\in J} Y_j$ (endowed with the product topology) and the projections $\mathrm{pr}_j \colon Y \to Y_j$. Then the natural map*

$$\Phi := (C(X, \mathrm{pr}_j))_{j\in J} \colon C(X, Y) \to \prod_{j\in J} C(X, Y_j)$$

is a homeomorphism.

Proof. The map Φ is a bijection (because a map $f \colon X \to Y$ is continuous if and only if all of its components $\mathrm{pr}_j \circ f$ are continuous). The topology on Y being initial with respect to the family $(\mathrm{pr}_j)_{j\in J}$, the topology on $C(X, Y)$ is initial with respect to the family $(C(X, \mathrm{pr}_j))_{j\in J}$ (Lemma B.4). Thus Φ is a topological embedding and hence a homeomorphism (being bijective). $\square$

It is often useful that $C(X, \varprojlim Y_j) = \varprojlim C(X, Y_j)$.

Lemma B.8 *Let $(J, \leq)$ be a directed set, $((Y_j)_{j\in J}, (\phi_{j,k})_{j\leq k})$ be a projective system of Hausdorff topological spaces*[14] *and Y be its projective limit, with the limit maps $\phi_j \colon Y \to Y_j$. Then the topological space $C(X, Y)$ is the projective limit of $((C(X, Y_j))_{j\in J}, (C(X, \phi_{j,k}))_{j\leq k})$, together with the limit maps $C(X, \phi_j) \colon C(X, Y) \to C(X, Y_j)$.*

[14]Thus $\phi_{j,k} \colon Y_k \to Y_j$ is a continuous map for $j, k \in J$ such that $j \leq k$, with $\phi_{j,j} = \mathrm{id}_{Y_j}$ and $\phi_{j,k} \circ \phi_{k,\ell} = \phi_{j,\ell}$ if $j \leq k \leq \ell$.

Proof. Let $P := \prod_{j\in J} Y_j$ and $\mathrm{pr}_j\colon P \to Y_j$ be the projection onto the j-th component. It is known that the map

$$\phi := (\phi_j)_{j\in J}\colon Y \to P$$

is a topological embedding with image $\phi(Y) = \{(x_j)_{j\in J} \in P\colon (\forall j,k \in J)\ j \leq k \Rightarrow x_j = \phi_{j,k}(x_k)\}$. Thus $C(X,\phi)$ is a topological embedding (see Lemma B.5) and hence so is

$$\Phi \circ C(X,\phi)\colon C(X,Y) \to \prod_{j\in J} C(X,Y_j),$$

using the homeomorphism $\Phi := (C(X,\mathrm{pr}_j))_{j\in J}\colon C(X,P) \to \prod_{j\in J} C(X,Y_j)$ from Lemma B.7. The image of $\Phi \circ C(X,\phi)$ is contained in the projective limit

$$L := \{(f_j)_{j\in J} \in \prod_{j\in J} C(X,Y_j)\colon (\forall j,k \in J)\, j \leq k \Rightarrow f_j = \phi_{j,k} \circ f_k\}$$

of the spaces $C(X,Y_j)$. If $(f_j)_{j\in J} \in L$ and $x \in X$, then $\phi_{j,k}(f_k(x)) = f_j(x)$ for all $j \leq k$, whence there exists $f(x) \in Y$ with $\phi_j(f(x)) = f_j(x)$. The topology on Y being initial with respect to the maps ϕ_j, we deduce from the continuity of the maps $\phi_j \circ f = f_j$ that $f\colon X \to Y$ is continuous. Now $(\Phi \circ C(X,\phi))(f) = (f_j)_{j\in J}$. Hence $\Phi \circ C(X,\phi)$ is a homeomorphism from $C(X,Y)$ onto L, and the assertions follow. $\square$

Also composition operators (or pullbacks) $C(f,Y)$ are essential tools.

Lemma B.9 *Let X_1, X_2 and Y be Hausdorff topological spaces. If a map $f\colon X_1 \to X_2$ is continuous, then also the map*

$$C(f,Y)\colon C(X_2,Y) \to C(X_1,Y), \quad \gamma \mapsto \gamma \circ f$$

is continuous.

Proof. If $K \subseteq X_1$ is compact and $U \subseteq Y$ an open set, then $f(K) \subseteq X_2$ is compact and $C(f,Y)^{-1}(\lfloor K,U \rfloor) = \lfloor f(K),U \rfloor$ (since $(\gamma \circ f)(K) \subseteq U \Leftrightarrow \gamma(f(K)) \subseteq U$ for $\gamma \in C(X_2,Y)$). $\square$

Remark B.10 If X_2 and Y are Hausdorff topological spaces and $X_1 \subseteq X_2$ a subset, then the restriction map

$$\rho\colon C(X_2, Y) \to C(X_1, Y), \quad \gamma \mapsto \gamma|_{X_1}$$

is continuous. In fact, since $\gamma|_{X_1} = \gamma \circ f$ with the continuous inclusion map $f\colon X_1 \to X_2$, $x \mapsto x$, we have $\rho = C(f, Y)$ and Lemma B.9 applies.

Lemma B.11 *Let X and Y be Hausdorff topological spaces and $(X_j)_{j\in J}$ be a family of subsets of X whose interiors X_j^0 cover X. Then the map*

$$\rho\colon C(X, Y) \to \prod_{j\in J} C(X_j, Y), \quad \gamma \mapsto (\gamma|_{X_j})_{j\in J}$$

is a topological embedding with closed image.

Proof. It is clear that the map ρ is injective, and it is continuous since each of its components is continuous, by the preceding remark. The image of ρ consists of all $(\gamma_j)_{j\in J} \in \prod_{j\in J} C(X_j, Y)$ such that

$$(\forall j, k \in J)\,(\forall x \in X_j \cap X_k) \quad \gamma_j(x) = \gamma_k(x)\,.$$

The point evaluation $\mathrm{ev}_x\colon C(X_j, Y) \to Y$ and the corresponding one on $C(X_k, Y)$ are continuous if $x \in X_j \cap X_k$. Since Y is Hausdorff (and thus the diagonal is closed in $Y \times Y$), it follows that $\mathrm{im}(\rho)$ is closed. Since ρ is injective, it will be an open map onto its image if it takes the elements of a subbasis to relatively open sets. To verify this property, let $K \subseteq X$ be compact and $U \subseteq Y$ be open. Each $x \in K$ is contained in the interior $X_{j_x}^0$ (relative X) for some $j_x \in J_0$. Because K is locally compact, x has a compact neighbourhood K_x in K such that $K_x \subseteq X_{j_x}^0$. By compactness, there is a finite subset $\Phi \subseteq K$ such that $K = \bigcup_{x\in\Phi} K_x$. Then $W_x := \lfloor K_x, U\rfloor \subseteq C(X_{j_x}, U)$ is an open subset of $C(X_{j_k}, Y)$ for all $x \in \Phi$. Hence $W := \{(\gamma_j)_{j\in J} \in \prod_{j\in J} C(X_j, Y)\colon (\forall x \in \Phi)\ \gamma_{j_x} \in W_x\}$ is open in $\prod_{j\in J} C(X_j, Y)$. Since $\rho(\lfloor K, U\rfloor) = W \cap \mathrm{im}(\rho)$, we see that $\rho(\lfloor K, U\rfloor)$ is relatively open in $\mathrm{im}(\rho)$, which completes the proof. □

Lemma B.12 *If X is a locally compact space and Y a Hausdorff topological space, then the evaluation map*

$$\varepsilon\colon C(X, Y) \times X \to Y, \quad (\gamma, x) \mapsto \gamma(x)$$

is continuous.

Proof. Let $U \subseteq Y$ be open and $(\gamma, x) \in \varepsilon^{-1}(U)$. By local compactness, there exists a compact neighbourhood $K \subseteq X$ of x such that $K \subseteq \gamma^{-1}(U)$. Then $\lfloor K, U \rfloor \times K$ is a neighbourhood of (γ, x) in $C(X, Y) \times X$ and $\lfloor K, U \rfloor \times K \subseteq \varepsilon^{-1}(U)$. As a consequence, $\varepsilon^{-1}(U)$ is open and thus ε is continuous. □

Lemma B.13 *Let X, Y and Z be Hausdorff topological spaces. If Y is locally compact, then the composition map*

$$\Gamma\colon C(Y, Z) \times C(X, Y) \to C(X, Z), \quad (\gamma, \eta) \mapsto \gamma \circ \eta$$

is continuous.

Proof. Let $K \subseteq X$ be compact and $U \subseteq Z$ be an open set. Let $(\gamma, \eta) \in \Gamma^{-1}(\lfloor K, U \rfloor)$. For each $x \in K$, there is a compact neighbourhood L_x of $\eta(x)$ in Y such that $L_x \subseteq \gamma^{-1}(U)$. We choose a compact neighbourhood M_x of $\eta(x)$ in Y such that $M_x \subseteq L_x^0$. Then $K_x := K \cap \eta^{-1}(M_x)$ is a compact neighbourhood of x in K. Hence, there is a finite subset $\Phi \subseteq K$ such that $K = \bigcup_{x \in \Phi} K_x$. Then $W := \bigcap_{x \in \Phi} \lfloor K_x, L_x^0 \rfloor$ is an open neighbourhood of η in $C(X, Y)$ and $V := \bigcap_{x \in \Phi} \lfloor L_x, U \rfloor$ is an open neighbourhood of γ in $C(Y, Z)$. We claim that $V \times W \subseteq \Gamma^{-1}(\lfloor K, U \rfloor)$. If this is true, then $\Gamma^{-1}(U)$ is open and hence Γ is continuous. To prove the claim, let $\sigma \in V$ and $\tau \in W$. If $y \in K$, then $y \in K_x$ for some $x \in \Phi$. Hence $\tau(y) \in \tau(K_x) \subseteq L_x^0$ and thus $\sigma(\tau(y)) \in \sigma(L_x) \subseteq U$, showing that $\sigma \circ \tau \in \lfloor K, U \rfloor$ indeed. □

Lemma B.14 *Let X and Y be Hausdorff topological spaces and $\mathcal{L}$ be a set of compact subsets of X. Assume that, for each compact subset $K \subseteq X$ and open subset $V \subseteq X$ with $K \subseteq V$, there exist $n \in \mathbb{N}$ and $K_1, \ldots, K_n \in \mathcal{L}$ such that $K \subseteq \bigcup_{i=1}^n K_i \subseteq V$. Then the sets*

$$\lfloor K, U \rfloor, \quad \textit{for } K \in \mathcal{L} \textit{ and open sets } U \subseteq Y,$$

form a basis for the compact-open topology on $C(X, Y)$.

Proof. Given a compact subset $K \subseteq X$ and open subset $U \subseteq Y$, let $\gamma \in \lfloor K, U \rfloor$. Then $V := \gamma^{-1}(U)$ is an open subset of X that contains K, whence we can find $K_1, \ldots, K_n \in \mathcal{L}$ as described in the hypotheses. Then $\gamma \in \bigcap_{i=1}^n \lfloor K_i, U \rfloor \subseteq \lfloor K, U \rfloor$. The assertion follows. □

Proposition B.15 *Let X, Y and Z be Hausdorff topological spaces. If $f\colon X \times Y \to Z$ is continuous, then also*

$$f^\vee\colon X \to C(Y,Z), \quad f^\vee(x) := f(x,\bullet) \tag{45}$$

is continuous. Moreover, the map

$$\Phi\colon C(X\times Y, Z) \to C(X, C(Y,Z)), \quad f \mapsto f^\vee \tag{46}$$

is a topological embedding. If Y is locally compact or $X\times Y$ is a k-space, then Φ is a homeomorphism.

Proof. Let $\gamma \in C(X\times Y, Z)$. To see that $\gamma^\vee$ is continuous, let $K \subseteq Y$ be compact and $U \subseteq Z$ be an open set. If $x \in X$ such that $\gamma^\vee(x) \in \lfloor K, U\rfloor$, then $\gamma(\{x\}\times K) \subseteq U$ and thus the product $\{x\}\times K$ of compact sets is contained in the open set $\gamma^{-1}(U)$. By the Wallace Lemma [16, 5.12], there is an open subset $V \subseteq X$ with $\{x\} \subseteq V$ and $V \times K \subseteq \gamma^{-1}(U)$. Then $V \subseteq (\gamma^\vee)^{-1}(\lfloor K, U\rfloor)$, showing that $(\gamma^\vee)^{-1}(\lfloor K,U\rfloor)$ is a neighbourhood of x and hence open (as x was arbitrary). Thus $\gamma^\vee$ is continuous.

To see that Φ is continuous, recall that the sets $\lfloor L, U\rfloor$, with $L \in \mathcal{K}(Y)$ and open sets $U \subseteq Z$, form a subbasis of the compact-open topology on $C(Y,Z)$. Hence, by Lemma B.2, the sets $\lfloor K, \lfloor L, U\rfloor\rfloor$ form a subbasis of the compact-open topology on $C(X, C(Y,Z))$, for L and U as before and $K \in \mathcal{K}(X)$. We claim that

$$\Phi^{-1}(\lfloor K, \lfloor L, U\rfloor\rfloor) = \lfloor K \times L, U\rfloor\,. \tag{47}$$

If this is true, then Φ is continuous. To prove the claim, let $\gamma \in C(X\times Y, Z)$. Then $\Phi(\gamma) \in \lfloor K, \lfloor L, U\rfloor\rfloor \Leftrightarrow \Phi(\gamma)(K) \subseteq \lfloor L, U\rfloor \Leftrightarrow (\forall x \in K)\ \gamma(x,\bullet) \in \lfloor L, U\rfloor$ $\Leftrightarrow (\forall x \in K)\ (\forall y \in L)\ \gamma(x,y) \in U \Leftrightarrow \gamma \in \lfloor K\times L, U\rfloor$, establishing (47).

Because Φ is (obviously) injective, it will be open onto its image if it takes open sets in a subbasis to relatively open sets. Let $\mathcal{L} \subseteq \mathcal{K}(X\times Y)$ be the set of all products $K \times L$, where $K \in \mathcal{K}(X)$ and $L \in \mathcal{K}(Y)$. We claim that $\mathcal{L}$ satisfies the hypotheses of Lemma B.14 (for the function space $C(X\times Y, Z)$). If this is so, then the sets $\lfloor K\times L, U\rfloor$ with $K \in \mathcal{K}(X)$ and $L \in \mathcal{K}(Y)$ form a subbasis for the compact-open topology on $C(X\times Y, Z)$, and now (47) shows that Φ is open onto its image (and hence a topological embedding).

To verify the claim, let $\pi_1\colon X\times Y \to X$ and $\pi_2\colon X\times Y \to Y$ be the projection onto the first and second component, respectively. If $M \subseteq X\times Y$ is compact

and $V \subseteq X \times Y$ an open subset such that $K \subseteq V$, then $M \subseteq M_1 \times M_2$, where $M_1 := \pi_1(M)$ and $M_2 := \pi_2(M)$ are compact. For each $(x, y) \in M$, there is an open neighbourhhod $U_{x,y} \subseteq X$ of x and and open neighbourhood $V_{x,y} \subseteq Y$ of y such that $U_{x,y} \times V_{x,y} \subseteq V$. Because M_1 and M_2 are locally compact, there exist compact neighbourhoods $K_{x,y} \subseteq M_1$ of x and $L_{x,y} \subseteq M_2$ of y such that $K_{x,y} \subseteq U_{x,y}$ and $L_{x,y} \subseteq V_{x,y}$. Then $K_{x,y} \times L_{x,y} \subseteq U_{x,y} \times V_{x,y} \subseteq V$. By compactness, K is covered by finitely many of the sets $K_{x,y} \times L_{x,y}$. Thus all hypotheses of Lemma B.14 are indeed satisfied.

Now assume that Y is locally compact. Let $\eta \in C(X, C(Y, Z))$. Because the evaluation map $\varepsilon \colon C(Y, Z) \times Y \to Z$ is continuous by Lemma B.12, the map $\gamma := \eta^\wedge := \varepsilon \circ (\eta \times \mathrm{id}_Y) \colon X \times Y \to Z$, $(x, y) \mapsto \eta(x)(y)$ is continuous. Since $\Phi(\gamma) = (\eta^\wedge)^\vee = \eta$, we see that Φ is surjective and hence (being an embedding) a homeomorphism.

Finally, assume that $X \times Y$ is a k-space. Again, we only need to show that Φ is surjective. Let $\eta \in C(X, C(Y, Z))$ and define $\gamma := \eta^\wedge \colon X \times Y \to Z$, $(x, y) \mapsto \eta(x)(y)$. If we can show that γ is continuous, then $\Phi(\gamma) = \eta$ (as required). Because $X \times Y$ is a k-space, it suffices to show that $\gamma|_K$ is continuous for each compact subset $K \subseteq X \times Y$. This will follow if $\gamma|_{K_1 \times K_2}$ is continuous for all compact subsets $K_1 \subseteq X$ and $K_2 \subseteq Y$ (given K as before, K is contained in the compact set $K_1 \times K_2$ with $K_1 := \pi_1(K)$, $K_2 := \pi_2(K)$). We now use that the restriction map $\rho \colon C(Y, Z) \to C(K_2, Z)$ is continuous (see Remark B.10) and hence also the map

$$\zeta := \rho \circ \eta|_{K_1} \colon K_1 \to C(K_2, Z).$$

Since K_2 is compact, $\zeta^\wedge \colon K_1 \times K_2 \to Z$ is continuous (by the preceding part of the proof). Because $\gamma|_{K_1 \times K_2} = \zeta^\wedge$, the continuity of γ follows. $\square$

Remark B.16 Given Hausdorff topological spaces X, Y, Z and a map $\eta \colon X \to C(Y, Z)$, define

$$\eta^\wedge \colon X \times Y \to Z, \quad (x, y) \mapsto \eta(x)(y).$$

The preceding proposition entails: If $\eta^\wedge$ is continuous, then $\eta = (\eta^\wedge)^\vee$ is continuous. If Y is locally compact or $X \times Y$ is a k-space, then $\eta = (\eta^\wedge)^\vee$ is continuous if and only if $\eta^\wedge$ is continuous.

Lemma B.17 *Let X and Y be Hausdorff topological spaces. For $y \in Y$, let $c_y\colon X \to Y$ be the constant map $x \mapsto y$. Then the map*

$$c\colon Y \to C(X,Y), \quad y \mapsto c_y$$

is continuous and (if $X \neq \emptyset$) in fact a topological embedding.

Proof. If $K \subseteq X$ is compact and $U \subseteq Y$ an open set, then $c^{-1}(\lfloor K,U \rfloor) = U$ (if $K \neq \emptyset$) and $c^{-1}(\lfloor K,U \rfloor) = Y$ (if $K = \emptyset$). In either case, the preimage is open, entailing that c is continuous. If X is not empty, we pick $x \in X$. Then the point evaluation $\mathrm{ev}_x\colon C(X,Y) \to Y$, $\gamma \mapsto \gamma(x)$ is continuous and $\mathrm{ev}_x \circ c = \mathrm{id}_Y$, entailing that c is a topological embedding. □

By the next lemma, so-called pushforwards are continuous.

Lemma B.18 *Let X, Y and Z be Hausdorff topological spaces and $f\colon X \times Y \to Z$ be continuous. Then also the following map is continuous:*

$$f_*\colon C(X,Y) \to C(X,Z), \quad \gamma \mapsto f \circ (\mathrm{id}_X, \gamma).$$

Thus $f_*(\gamma)(x) = f(x, \gamma(x))$.

Proof. Identifying $C(X,X) \times C(X,Y)$ with $C(X, X \times Y)$ as in Lemm B.7, we can write $f_*(\gamma) = C(X,f)(\mathrm{id}_X, \gamma)$. Since $C(X,f)$ is continuous by Lemma B.3, the continuity of f_* follows. □

We also have two versions with parameters.

Lemma B.19 *Let X, Y, Z and P be Hausdorff topological spaces and $f\colon P \times X \times Y \to Z$ be a continuous map. For $p \in P$, abbreviate $f_p := f(p, \bullet)\colon X \times Y \to Z$. Then also the following map is continuous:*

$$P \times C(X,Y) \to C(X,Z), \quad (p,\gamma) \mapsto f_p \circ (\mathrm{id}_X, \gamma).$$

Proof. Let $c\colon P \to C(X,P)$, $p \mapsto c_p$ be the continuous map discussed in Lemma B.17. The map $C(X,f)\colon C(X, P \times X \times Y) \to C(X,Z)$ is continuous (by Lemma B.3). Identifying $C(X,P) \times C(X,X) \times C(X,Y)$ with $C(X, P \times X \times Y)$, we have $f_p \circ (\mathrm{id}_X, \gamma) = C(X,f)(c_p, \mathrm{id}_X, \gamma)$, which is continuous in (p,γ). □

Lemma B.20 *Let X, Y, Z and P be Hausdorff topological spaces and $f\colon P \times Y \to Z$ be a continuous map. For $p \in P$, set $f_p := f(p, \bullet)\colon Y \to Z$. Then also the following map is continuous:*

$$\psi\colon P \times C(X,Y) \to C(X,Z), \quad (p,\gamma) \mapsto f_p \circ \gamma. \tag{48}$$

Proof. Since $g\colon P \times X \times Y \to Z$, $g(p,x,y) := f(p,y)$ is continuous, so is $\psi\colon P \times C(X,Y) \to C(X,Z)$, $(p,\gamma) \mapsto g_p \circ (\mathrm{id}_X, \gamma)$, by Lemma B.19. $\square$

Lemma B.21 *If X is a Hausdorff topological space and G a topological group, then the following holds:*

(a) *$C(X,G)$ is a topological group with respect to the pointwise group operations.*

(b) *Let $\mathcal{L}$ be a set of compact subsets of X such that each $K \in \mathcal{K}(X)$ is contained in some $L \in \mathcal{L}$. Also, let $\mathcal{U}$ be a basis of open identity neighbourhoods in G. For $\gamma \in C(X,G)$, the sets $\gamma \cdot \lfloor K, U \rfloor$ then form a basis of open neighbourhood of γ in $C(X,G)$ (for $K \in \mathcal{L}$ and $U \in \mathcal{U}$ in a basis $\mathcal{U}$), and so do the sets $\lfloor K, U \rfloor \cdot \gamma$. The compact-open topology therefore coincides with the topology of uniform convergence on compact sets, both with respect to the left and also the right uniformity on G.*

(c) *If X is hemicompact and G is metrizable, then $C(X,G)$ is metrizable.*

(d) *If X is a k-space and G is complete, then $C(X,G)$ is complete.*

(e) *If X is a k-space and G is sequentially complete, then $C(X,G)$ is sequentially complete.*

(f) *If E is a topological vector space over a topological field $\mathbb{K}$, then the pointwise operations make $C(X,E)$ a topological $\mathbb{K}$-vector space. Moreover, $C(X,\mathbb{K})$ is a topological $\mathbb{K}$-algebra and $C(X,E)$ is a topological $C(X,\mathbb{K})$-module.*

(g) *If $(\mathbb{K}, |.|)$ is an ultrametric field and E a locally convex topological $\mathbb{K}$-vector space, then also $C(X,E)$ is locally convex.*

(h) *If $(\mathbb{K}, |.|)$ is an ultrametric field, E an ultrametric normed space and X is compact, then also $C(X,E)$ admits an ultrametric norm defining its topology.*

Proof. (a) Since G is a topological group, the group multiplication $\mu\colon G\times G\to G$, $(x,y)\mapsto xy$ and the inversion map $\iota\colon G\to G$, $x\mapsto x^{-1}$ are continuous. Identifying $C(X,G)\times C(X,G)$ with $C(X,G\times G)$ (as in Lemma B.7), the group multiplication of $C(X,G)$ is the mapping $C(X,\mu)\colon C(X,G\times G)\to C(X,G)$, which is continuous by Lemma B.3. The group inversion is the map $C(X,\iota)\colon C(X,G)\to C(X,G)$ and hence continuous as well. Thus $C(X,G)$ is a topological group.

(b) If $K_1,\ldots,K_n\in\mathcal{K}(X)$ and $U_1,\ldots,U_n\subseteq G$ are open identity neighbourhoods, we find $U\in\mathcal{U}$ such that $U\subseteq U_1\cap\cdots\cap U_n$, and $K\in\mathcal{L}$ such that $K_1\cup\cdots\cup K_n\subseteq K$. Then $\bigcap_{j=1}^n\lfloor K_j,U_j\rfloor\supseteq\lfloor K,U\rfloor$. Therefore the sets $\lfloor K,U\rfloor$ with $K\in\mathcal{L}$ and $U\in\mathcal{U}$ form a basis of open identity neighbourhoods for $C(X,G)$. Since left and right translations in the topological group $C(X,G)$ are homeomorphisms, the remainder of (b) follows.

(c) Let $K_1\subseteq K_2\subseteq\cdots$ be an ascending sequence of compact subsets of X, with union X, such that each compact subset of X is contained in some K_n. Also, let $U_1\supseteq U_2\supseteq\cdots$ be a descending sequence of open identity neighbourhoods in G which gives a basis for the filter of identity neighbourhoods. By (b), the sets $\lfloor K_n,U_n\rfloor$, for $n\in\mathbb{N}$, provide a countable basis of identity neighbourhoods in $C(X,G)$. As a consequence, the topological group $C(X,G)$ is metrizable [15, Theorem 8.3].

(d) Let $(\gamma_a)_{a\in A}$ be a Cauchy net in $C(X,G)$. For each $x\in X$, the point evaluation $\mathrm{ev}_x\colon C(X,G)\to G$ is a continuous homomorphism. Hence $(\gamma_a(x))_{a\in A}$ is a Cauchy net in G and hence convergent to some element $\gamma(x)\in G$ (as G is a assumed complete). For each compact set $K\subseteq X$, the restriction map $\rho_K\colon C(X,G)\to C(K,G)$ is a continuous homomorphism (cf. Remark B.10), whence $(\gamma_a|_K)_{a\in A}$ is a Cauchy net in $C(K,G)$. We claim that $C(K,G)$ is complete. If this is true, then $\gamma_a|_K\to\gamma_K$ for some continuous map $\gamma_K\in C(K,G)$. Since $\gamma(x)=\gamma_K(x)$ for each $x\in K$, we see that $\gamma|_K=\gamma_K$ is continuous. Because X is a k-space, this implies that γ is continuous. If $K\in\mathcal{K}(X)$ and $U\subseteq G$ is an identity neighbourhood, then $(\gamma_K)^{-1}\gamma_a|_K\in\lfloor K,U\rfloor$ inside $C(K,G)$ for sufficiently large a (see (b)) and hence $\gamma^{-1}\gamma_a\in\lfloor K,U\rfloor$ inside $C(X,G)$, showing that $\gamma_a\to\gamma$.

To prove the claim, we may assume that $X=K$ is compact. Let $U\subseteq G$ be an open identity neighbourhood and $V\subseteq U$ be an open identity neighbourhood with closure $\overline{V}\subseteq U$. There is $a\in A$ such that $\gamma_b^{-1}\gamma_c\in\lfloor K,V\rfloor$ for all $b,c\geq a$ in A. Thus, for all $x\in K$, $\gamma_b(x)^{-1}\gamma_c(x)\in V$. Passing to the limit

in c, we obtain $\gamma_b(x)^{-1}\gamma(x) \in \overline{V} \subseteq U$ and thus

$$\gamma(x) \in \gamma_b(x)U, \quad \text{for all } x \in K \text{ and } b \geq a. \tag{49}$$

If W is any identity neighbourhood in G, we can choose U from before so small that $U^{-1}UU \subseteq W$. By continuity of γ_a, each $x \in K$ has a neighbourhood $L \subseteq K$ such that $\gamma_a(y)^{-1}\gamma_a(y) \in U$ for all $y \in L$. Combining this with (49), we see that

$$\gamma^{-1}(y)\gamma(x) \in U^{-1}\gamma_a(y)^{-1}\gamma_a(x)U \subseteq U^{-1}UU \subseteq W$$

for all $y \in L$. Thus γ is continuous at x and hence continuous. Finally, we have $\gamma_b^{-1}\gamma \in \lfloor K, U \rfloor$ for all $b \geq a$, by (49). Hence $\gamma_b \to \gamma$ in $C(K,G)$.

(e) Proceed as in (d), replacing the Cauchy net with a Cauchy sequence.

(f) By (a), $C(X,E)$ is a topological group. Let $\sigma\colon \mathbb{K}\times E \to E$, $(z,v) \mapsto zv$ be multiplication with scalars. Then the map $\mathbb{K} \times C(X,E) \to C(X,E)$, $(z,\gamma) \mapsto \sigma(z,\bullet) \circ \gamma$ is continuous (by Lemma B.20). As this is the multiplication by scalars in $C(X,E)$, the latter is a topological vector space. Let $\mu\colon \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ be the multiplication in the field $\mathbb{K}$. Identifying $C(X,\mathbb{K}) \times C(X,\mathbb{K})$ with $C(X,\mathbb{K}\times\mathbb{K})$, the algebra multiplication of $C(X,\mathbb{K})$ is the map $C(X,\mu)\colon C(X,\mathbb{K}\times\mathbb{K}) \to C(X,\mathbb{K})$, which is continuous by Lemma B.3. Hence $C(X,\mathbb{K})$ is a topological algebra. Identifying $C(X,\mathbb{K})\times C(X,E)$ with $C(X,\mathbb{K}\times E)$, the $C(X,\mathbb{K})$-module multiplication on $C(X,E)$ is the map $C(X,\sigma)\colon C(X,\mathbb{K}\times E) \to C(X,E)$, which is continuous by Lemma B.3. Hence $C(X,E)$ is a topological $C(X,\mathbb{K})$-module.

(g) Let $\mathbb{O} := \{z \in \mathbb{K}\colon |z| \leq 1\}$ be the valuation ring of $\mathbb{K}$. By hypothesis, the open $\mathbb{O}$-submodules $U \subseteq E$ form a basis of 0-neighbourhoods in E. For each $K \in \mathcal{K}(X)$, the set $\lfloor K, U \rfloor$ then is an open 0-neighbourhood in $C(X,E)$ and an $\mathbb{O}$-module. Part (b) implies that these sets form a basis of 0-neighbourhoods in $C(X,E)$. Thus $C(X,E)$ is locally convex.

(h) If the ultrametric norm $\|.\|$ defines the topology of E, then the ultrametric norm $\|.\|_\infty\colon C(X,E) \to [0,\infty[$, $\|\gamma\|_\infty := \sup\{\|\gamma(x)\|\colon x \in X\}$ defines the topology of $C(X,E)$, as is clear from (b) and the observation that $\lfloor X, B_r^E(0)\rfloor = B_r^{C(X,E)}(0)$ for each $r > 0$. $\square$

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Helge Glöckner, Universität Paderborn, Institut für Mathematik,
Warburger Str. 100, 33098 Paderborn, Germany. Email: `glockner@math.upb.de`